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PROCEEDINGS
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INTERNAL MOTION IN THE SPIRAL NEBULA MESSIER 33

PRELIMINARY RESULTS

BY ADRIAAN VAN MAANEN

MOUNT WILSON OBSERVATORY, CARNEGIE INSTITUTION OF WASHINGTON

Communicated by G. E. Hale, November 10, 1920

One of the most interesting questions in astronomy concerns the relation of the spiral nebulae to the galactic system. Are they or are they not separate stellar systems comparable with our own? Especially during the last few years have many arguments been advanced both for and against. Real internal motions of the order indicated by the measures on Messier 101, communicated to these PROCEEDINGS in 1916,[1] would raise a strong objection to the "island-universe" hypothesis. If comparable with our own system, spiral nebulae would be so distant that no ordinary velocities of their constituent parts would yield measurable displacements in the short interval of a few years.

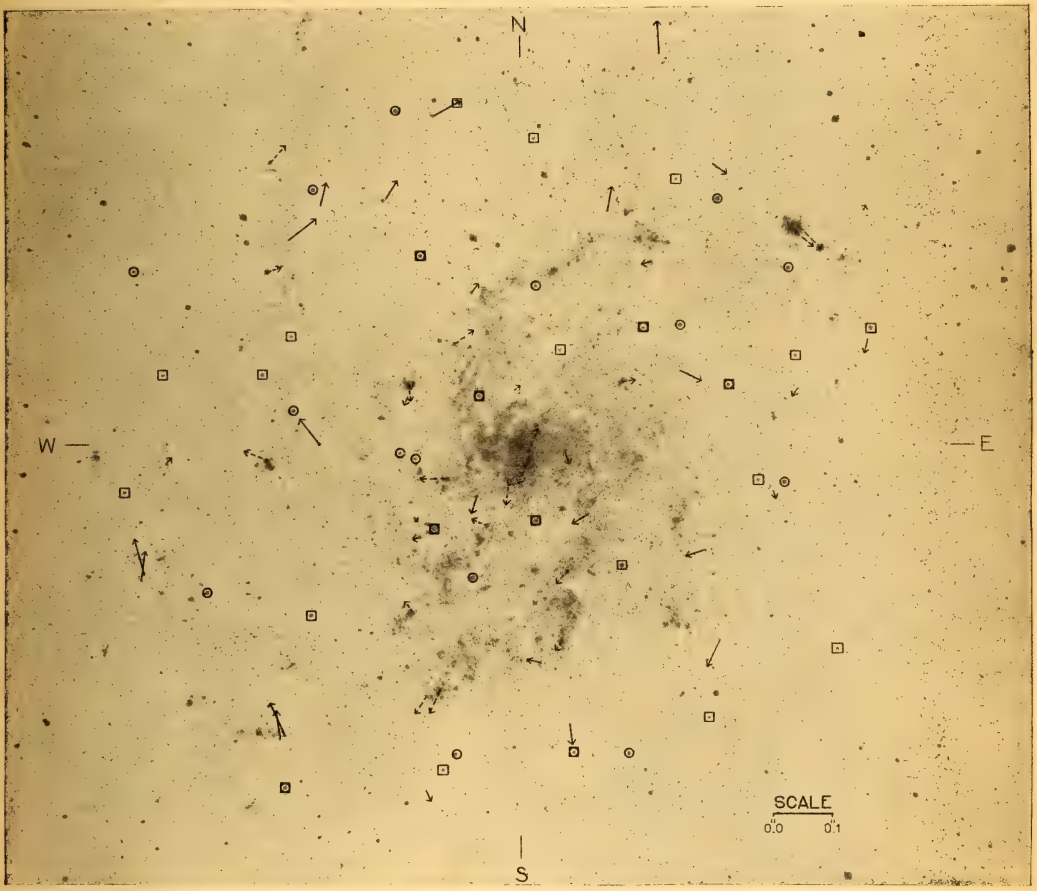
The measures on Messier 101 were, however, based principally on two plates taken by Mr. Ritchey at the 25-foot focus of the 60-inch reflector with an interval of only five years, and required confirmation before a general acceptance of the results could be expected. This was afforded in part by measures of three photographs of the same object taken at the Lick Observatory, which were kindly loaned by Director Campbell. Measures on Messier 81 also pointed in the same direction; while Lampland[2] found analogous motions for Messier 51 in 1916, and Kostinsky[3] in 1917.

Meanwhile another five years has elapsed, which would have doubled the interval for the 60-inch reflector photographs of Messier 101; but, unfortunately, bad weather during the spring made it impossible to obtain a satisfactory plate. In August, 1920, however, Mr. Duncan secured a fairly good photograph of another spiral, Messier 33, for which an excellent plate, taken by Mr. Ritchey in 1910, was also available. Messier 33 is, moreover, better adapted for such measures than Messier 101, on account of its numerous condensations and their close approach to starlike appearance. For this same object two plates of short exposure, taken at the 80-foot focus of the 60-inch reflector in 1915 and 1920, were also available.

Both these pairs of plates have been measured with the monocular arrangement of the Zeiss stereocomparator. It has been shown that with temperature properly controlled this instrument is admirably adapted to such measures, because it allows the bisection in quick succession of corresponding points on the two photographs; the possibility of not bisecting a nebulous and asymmetrical point in the same way on both plates is thereby greatly reduced, if not wholly avoided. A note on the unreliability of the stereocomparator for accurate measures has recently been published by Kreiken;[4] but, as I have pointed out several times it is only necessary to keep the temperature of the room constant in order to secure good results with the Zeiss instrument. In our case the change in temperature during a series of measures can be kept within a few tenths of a degree. The influence of this variation is extremely small; but, as an extra precaution, we have adopted the plan of measuring the n points, first in the order 1 . . . . . . n, then in the order n . . . . . . 1. In this way practically all temperature effect can be eliminated, and it is easy to prove that the measures of the present series, which are of moderate length, cannot be affected by more than 10 or 12 per cent. For a series of several hundred points, whose measurement would take several days, the conditions are not so favorable, and in the present case it was thought best to await the completion of a new stereocomparator, now under construction in the instrument shop of the Observatory, in which the principal sources of trouble have been avoided, before undertaking the complete measurement of the plates.

For a preliminary discussion, 24 comparison stars and 23 nebulous points were selected on the plates taken at the 25-foot focus. When the measures had been finished, Mr. Seares was kind enough to call my attention to some photographs of Messier 33 taken by him with and without color-screens. A few years ago Seares found that the outer points of spiral nebulae are decidedly blue in color. A rough comparison of his plates showed that I had been fortunate in the choice of both comparison stars and nebular points. Of the first only three are blue, while of the latter all but two are decidedly blue; these two points have been omitted in the following discussion, since they may not belong to the nebula.

On the plates taken by me at the 80-foot focus all measurable points, a total of 89, were used. After the measures were finished I selected, with the help of the photographs taken by Mr. Ritchey and by Mr. Seares, 23 objects as comparison stars; these are all starlike in appearance and reddish in color. Ten points which are either very bright or very faint or for which the measures are uncertain were excluded. Of the remaining points, 22, including the centre, almost certainly belong to the nebula, as they seem to be involved in the nebular knots and, with the exception of the centre, are decidedly blue. Fourteen points are probably nebular, being seemingly involved in nebulosity but not decidedly blue, or blue but not so clearly involved in the nebula. For 20 points it is quite uncertain whether or not they form part of the nebula.



Internal motions in the spiral nebula Messier 33. The dotted- and full-line arrows show the motions of points in the nebula derived from each of two pairs of photographs. The comparison stars are enclosed in small squares and circles.

For each set of plates the proper motions μ_α and μ_δ were derived for all the points measured with respect to the 24 and 23 comparison stars used. These motions for the points belonging to the nebula are presumably due partly to a translation of the nebula as a whole, and partly to internal motion. By a method analogous to that followed for Messier 101 the motion of translation of the nebula was found to be

$$\mu_\alpha = -0''.007, \ \mu_\delta = -0''.002.$$

The results found by subtracting these values from the motions of the nebular points are assumed to represent internal motions. In plate I those for the 30 points of the 25-foot focus plates are indicated by arrows with full lines and for the 22 very probable nebular points measured on the 80-foot focus plates by arrows with broken lines; the comparison stars of the first pair are enclosed in small squares, those of the second pair in circles.

A first inspection of the plate leaves us again in doubt as to whether we are dealing with a rotation of the nebula as a whole or with a motion along the arms of the spiral; upon further examination the latter motion seems to be more clearly indicated.

Resolving the motions into components perpendicular to and along the radius, we find the following results:

	25-FOOT FOCUS	80-FOOT FOCUS
Right-handed	23 or 77%	17 or 81%
Left-handed	7 or 23%	4 or 19%
Outward	18 or 60%	15 or 71%
Inward	12 or 40%	6 or 29%
Mean tangential motion	+0″.020	+0″.014
Mean radial motion	+0″.006	+0″.012

+ means right-handed and outward

Classifying the points according as they have a component of motion outward or inward along the arms of the spiral, we have the following results:

	25-FOOT FOCUS	80-FOOT FOCUS
Motion outward	28 or 93%	18 or 86%
Motion inward	2 or 7%	3 or 14%
Mean outward motion	0″.024	0″.021

Before discussing the possibility of an increase or decrease of motion with distance from the centre it will be necessary to measure a large number of additional points in the nebula, which is deferred for the present. The principal conclusion to be drawn from the present material is that here again we find motions analogous to those occurring in Messier 101, 81 and 51. In general they seem to be outward along the arms of the spiral.

In 1916 Pease published values for the radial velocity of the centre of Messier 33 and for the bright knot 10′ *nf* the nucleus. The two values differ by about 200 km./sec. Taking into account the probable inclination of the nebula with respect to the tangential plane, we can gain some idea of the order of the parallax of the nebula by comparing Pease's results with those obtained from the present investigation; the corresponding parallax is about 0″.0005. The diameter of the nebula would be about 100 light-years and the individual points of the nebula would have absolute magnitudes of +1 and fainter.

If on the other hand we suppose the dimensions of Messier 33 to be comparable with those of the galactic system, its distance would be several million light-years; the motions indicated by the photographs would then represent velocities of the order of 150,000 to 300,000 km./sec., which, obviously, are extremely improbable. The internal motions in the spirals seem now to be well founded, and if time justifies this belief, they will accordingly afford a most important argument against the view that these nebulae are systems comparable with our galaxy.

[1] van Maanen, A., *Mt. Wilson Contr.*, No. 118, 1916; *Astrophys. J., Chicago, Ill.*, **44,** 1916 (210–228).

[2] *Amer. Astron. Soc. Publ.*, **3,** 1918 (206–207).

[3] *Monthly Notices, Lond.*, **77,** 1917 (233–234).

[4] *Observatory, Lond.*, **43,** 1920 (255–260).

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*THE ATTACHMENT OF ELECTRONS TO NEUTRAL MOLECULES IN AIR**

BY LEONARD B. LOEB†

RYERSON PHYSICAL LABORATORY, UNIVERSITY OF CHICAGO

Communicated by R. A. Millikan, December 4, 1920

It is well known that the process of ionization in gases consists of the detachment of an electron from the molecules or atoms ionized. It has further been proved that in air at atmospheric pressure the carriers of negative electricity are neutral molecules[5,6] of the gas carrying an additional electron. These are called the normal ions. Now it is of interest to determine in what manner the electron liberated by the ionizing process attaches itself to a molecule to form an ion. For it is possible that an understanding of this process may help us to gain a picture of the surfaces of the molecules.

In air at atmospheric pressure the normal negative ions move with a velocity of 2 cm./sec. in unit electric field while the electrons have a mobility of about 200 cm./sec.[3,7] under the same conditions. Such a marked difference in the two types of carriers accordingly furnishes us an excellent means of investigating the above question as the ensuing analysis will show.

Let us assume that the electrons liberated from the plate P, figure 1, by ultraviolet light unite with gas molecules on their first impacts to form ions. Now let a second plate, E, connected to an electrometer be placed parallel to P and d cm. above it. On applying an alternating potential difference of half period T between P and E the ions starting from P will alternately be driven from, and drawn back to P, for periods of T seconds. If u is the velocity of the ion in unit field, then for given values of T and d there will be no current to E until V reaches a value V_0 such

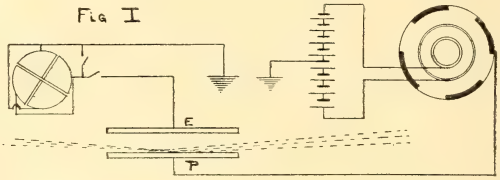
Fig I

that $uTV_0/d = d$. At V_0 then a current to E will become manifest which will increase with increasing V to a saturation value. On plotting the current i as ordinate against the voltage V as abscissa we will get a curve of the type of curve I, figure II. Such a curve will be termed a mobility curve for if the value of the intercept V_0 of such an experimental curve be substituted in the equation

$$u = d^2/V_0T \qquad (1)$$

the mobility u of the carrier is at once obtained if d and T are known. When the pressures are reduced the curves remain the same but the value of the mobility u of the ion is found to increase inversely as the pressure. In other words $up/760 = K$, where K, the *mobility constant* of the ions, is independent of pressure.

Now let us assume with J. J. Thomson[1] that an electron does not attach to form an ion on its first encounter with a gas molecule, but that the attachment is a chance phenomenon occurring on the average in one out of n impacts with molecules. Let us further assume that for air n is large, say, about 250,000. Under ordinary conditions the velocity of thermal agitation of the electron is great compared with its velocity acquired in the electric field. The electron, therefore, covers a zigzag path drifting slowly all the while in the direction of the electric field. Consequently while making the 250,000 impacts necessary for attachment the majority of the electrons will move a distance, A, in the field, which may be estimated. Since A is covered by the electron in 1/100 of the time that an ion would take, one may practically assume that the majority of the ions start at a point distant A cm from P instead of from P. For appreciable values of A the values of V required to drive the majority of the carriers to E may become notably lower than they are for ions starting at P. The values of u and hence of K obtained from experiments where A becomes significant will then be high. In air at atmospheric pressure the time taken for an electron to make 250,000 impacts is about 1.2×10^{-6} sec. In a field of 40 volts with $d = 15$ mm. the electron would cover a distance of 6.3×10^{-2} mm. towards E in this time. This small value of the distance A covered as an electron would produce no measurable effect on $V_\circ$ and the values of K would be normal. Now let the pressure be reduced to 152 mm. The mobility of the electrons will be increased five-fold. Furthermore as the density of the gas is reduced to one-fifth the electrons will take five times as long to make the 250,000 impacts required to attach. The distance A will, therefore, become 1.5 mm. which is an appreciable fraction of the 15.0 mm. between the plates. The intercept $V_\circ$ as determined from the portion of the curve corresponding to the majority of the carriers will, therefore, be but 0.9 of the $V_\circ$ required for ions, and the value of K will be about 10% higher than for normal ions. In other words assuming the Thomson mode of attachment the mobility constant of the ions will appear to increase with decreasing pressures below about 152 mm.[2] Since the number of impacts required to form an attachment is a *chance* phenomenon there will be an appreciable number of ions reaching E that have traversed distances greater than A before attaching. These will reach E at values of $V_\circ$ below those for the majority of carriers. As a result the otherwise sharp intercepts of mobility curves with the voltage axis observed for *ions* will be masked by asymptotic feet of exponential form rising at values of V well below the $V_\circ$ required for the majority of the ions. These will become rapidly more pronounced as the pressures decrease. Such curves are seen in the experimental curves II, III, IV, V and VI, figure II, obtained in air under conditions indicated in the legend. To estimate the *hybrid* mobilities of such carriers the asymptotic feet must naturally be ignored, as has been done by previous observers.[2]§

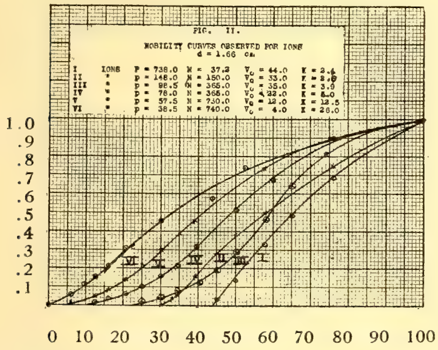
FIG. II.

MOBILITY CURVES OBSERVED FOR IONS
d = 1.66 cm.

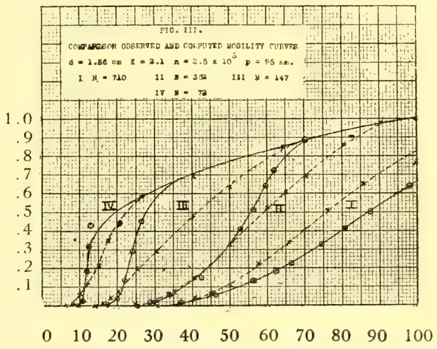
FIG. III.

COMPARISON OBSERVED AND COMPUTED MOBILITY CURVES

With the method of analysis outlined above it would be possible to pursue the evolution of the mobility curves as a function of p and T in a qualitative manner indefinitely. Fortunately a much simpler and more accurate study of the theory is made possible by the application of the mathematical analysis of Thomson. On the basis of his theory Thomson has shown that out of $Q_\circ$ electrons starting from P the number that can travel x cm. through the air without combining to form ions is given by

$$Q/Q_\circ = e^{-(Wx)/(nK'LV/d)} \tag{2}$$

where W is the velocity of thermal agitation of the electron, K' is the mobility of the electron, L is its mean free path, V is the voltage, d is the distance between the plates, and n is the chance of ion formation per impact.

Let us assume that when the voltage V is equal to $V_\circ$ *all* the $Q_\circ$ carriers starting from P at atmospheric pressures can reach E. This simplifying assumption is contrary to fact for at $V_\circ$ the carriers actually only *begin* to reach E. We may with this assumption impose the conditions implied in equation (1) on the equation (2). This will lead to the conclusion that out of a maximum possible current $I_\circ$, the real current I which reaches E as a function of the frequency $N = 1/T$, the pressure p, the plate distance d, and the voltage V, is given by the equation

$$I = I_\circ e^{-\frac{W}{nK'L}\left(\frac{d^2(p/760)^2}{V} - \frac{K(p/760)}{N}\right)} \tag{3}$$

where p is in mm. of mercury, and K is the mobility of the normal ion. Now it is possible to evaluate W from the mean kinetic energy of the molecules, for it is assumed in the theory that the electrons move in the electric field with a velocity small compared to W. Let us further assume K' to be constant and equal to 200 cm./sec.,[7,3] while we take L as $4\sqrt{2}$ times the mean free path of the molecules, and K as about 2.5. We thus have the equation

$$I = I_\circ e^{-\frac{9.9 \times 10^8}{n}\left(\frac{d^2(p/760)^2}{V} - \frac{(2.5p/760)}{N}\right)} \tag{4}$$

This equation is open to experimental verification for it contains but one unknown quantity n, as $I/I_\circ$ can be determined experimentally under known conditions of N, p, V and d.

I have recently made a series of determinations of the mobilities of the carriers produced in air at different pressures under essentially the simple conditions in the foregoing discussion. These determinations yielded mobility curves of which the set of curves shown in figure II are typical. As is seen at once the form of the curves resembles the curves to be expected from the qualitative discussion above. The values of $I/I_\circ$ may be determined in such curves from the ratio of the current to E caused by a given alternating potential between P and E and that caused by an equal fixed negative potential on P. By substituting this value of $I/I_\circ$ in the equation with the corresponding values of N, P, V and d one may solve for n. As the result of a large number of determinations the value of n obtained under conditions best conforming to the theoretical assumptions was 250,000. With n determined the theory may be further tested by computing the curves for $I/I_\circ$ as a function of V for different values of d, N and p. A large number of curves were thus computed. A typical comparison of the theoretical curves so obtained and the curves actually observed under the same conditions may be seen in figures III, IV and V.

The dotted curves are the observed and the full curves are the computed ones. The experimental data are given in the legend. The shapes of the curves at atmospheric pressures or at low values of N, where the carriers are all ions, should not agree. For in deducing the theory we assumed that at V_0 *all* carriers succeeded in reaching E, which is not the case in fact. Also at the lower pressures close agreement is not to be expected, for the velocity acquired in the field becomes commensurable with the velocity of agitation W. Barring these points the general shapes of the theoretical and observed curves, and the changes in shape of these curves with p and N, are quite similar. It is also evident that the points of inflection, or the asymptotic feet, of the corresponding observed and computed curves fall on values of V which lie close together. This means that the values of the hybrid, or abnormal, mobilities estimated from the two sets of curves are nearly the same. There is, therefore, sufficient similarity in the two sets of curves to permit one to assert that the curves are represented by the same type of equation. We may then conclude that the Thomson theory is in *good qualitative* agreement with the results observed.

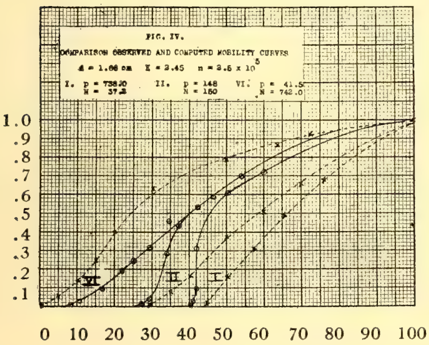
FIG. IV.

COMPARISON OBSERVED AND COMPUTED MOBILITY CURVES

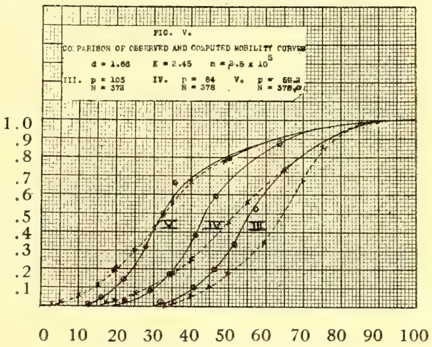
FIG. V.

COMPARISON OF OBSERVED AND COMPUTED MOBILITY CURVES

With the evidence before us I believe we are justified in concluding that the mechanism of negative ion formation in air consists in the electron attaching itself to a molecule to form a negative ion on the average in one out of 250,000** molecular impacts. Now it has been shown that in pure nitrogen[3,4] the electron does not attach to form an ion to any appreciable extent. We must, therefore, assume that the electron attaches to the oxygen molecule in air. Measurements made in pure oxygen give n as about 50,000. As there are four molecules of nitrogen in air to one of oxygen this agrees quite well with our conclusion. What the significance of n is, whether it depends on the electronic ring in the molecule struck, whether it depends on some particular state of the molecule struck, or whether it depends on the energy conditions of the impact remains for the future to say.

In conclusion I desire to express my thanks to Professor R. A. Millikan for his kind advice and criticism.

* A more detailed description of these experiments will later appear in the *Physical Review.*

† NATIONAL RESEARCH FELLOW of the NATIONAL RESEARCH COUNCIL.

§ Such mobilities have been obtained below 100 mm pressures in air by all the experimenters in this field up to the time of Wellisch. Until Thomson proposed his theory they had never been adequately explained.

** This value depends on the correctness of the assumptions as to the numerical values of K' and L for the electron.

[1] Thomson, J. J., *London Phil. Mag.*, Sept., 1915.

[2] Kovarik, A. F., *Physic. Rev., Ithaca,* **30,** 1910 (415).

[3] Franck, J., *Verh. deuts. physik. Ges.*, **12,** 1910 (613).

[4] Loeb, L. B., These PROCEEDINGS, June, 1920.

[5] Loeb, L. B., *Physic. Rev.*, **8,** 1916 (6).

[6] Yen, K. L., *Ibid.*, **9,** 1918 (5).

[7] Wellisch, E. M., *Amer. J. Sci., New Haven,* (Ser. 4) **44,** 1917 (1); *Phil. Mag.*, **34,** July, 1917.

THE BASAL METABOLISM OF GIRLS 12 TO 17 YEARS OF AGE

BY FRANCIS G. BENEDICT, MARY F. HENDRY AND MARION L. BAKER

NUTRITION LABORATORY, CARNEGIE INSTITUTION OF WASHINGTON, BOSTON

Read before the Academy, November 17, 1920

The Nutrition Laboratory's task of charting the field of basal metabolism of humans from birth to old age has resulted in a reasonable completion of the study of boys and girls from birth to puberty, of both sexes, from the college age to 35 years, and of women about 50 years of age. The metabolism during the important age-range from 12 to 17 years, representing as it does a period of rapid growth as well as the period of the establishment of puberty, has recently been studied, so far as girls are concerned, in a large respiration chamber permitting the simultaneous measurement of the carbon-dioxide production of a dozen or more subjects. Groups of twelve Girl Scouts each volunteered as subjects, and a typical experiment involved their entering the respiration chamber after a light standard supper and sleeping quietly throughout the night. The entire carbon-dioxide production during the period of "bed rest," as well as the "minimum" carbon-dioxide production found throughout the night, are the bases for the computations of the energy needs for "bed rest" and for the basal metabolism. The quiet, resting morning pulse rate was obtained with all groups and the insensible perspiration was also measured with most of the groups.

Special interest centres around the gaseous metabolism or, more particularly, the energy computations therefrom. It was found that 55.0 calories represented the average hourly heat production per individual, with very little difference due to either average age or average weight.

For groups of girls, therefore, from 12 to 17 years of age the energy requirement for 10 hours of "bed rest" may be taken as 550 calories. The basal 24-hour heat production with all these groups of girls closely approximated 1250 calories per individual, irrespective of age or weight.

The computation of the heat per kilogram of body weight per 24 hours, the classic method of equalizing, in so far as possible, differences in the weights of the groups, showed that in general the younger the average age, the higher the heat production. Similarly, computations of the heat production per square meter of body surface (obtained from the Du Bois height-weight chart) indicated a distinct tendency for the higher values to be found with the younger girls. These data for the basal metabolism measurements are given in table 1 herewith.

TABLE 1

BASAL HEAT PRODUCTION OF GIRLS 12 TO 17 YEARS OF AGE

DATE[1]		AVERAGE AGE		AVERAGE BODY WEIGHT WITHOUT CLOTHING	AVERAGE HEIGHT	BASAL HEAT (COMPUTED)[2] PER 24 HOURS		
						Per individual	Per kilo	Per sq. m.
1920		Yrs.	Mo.	Kgs.	Cms.	Cals.	Cals.	Cals.
Jan.	2– 3	17	0	58.1	163	1258	21.7	777
Jan.	9–10	15	10	53.6	163	1170	21.8	745
Jan.	16–17	14	10	49.9	160	1254	25.1	830
April	9–10	14	1	44.2	155	1266	28.6	911
March	19–20	14	0	51.7	161	1331	25.7	876
Jan.	23–24	13	8	50.0	159	1382	27.6	928
Feb.	13–14	13	4	43.7	153	1188	27.2	867
March	12–13	12	10	41.0	152	1226	29.9	922
March	5– 6	12	2	39.5	150	1180	29.9	915

[1] Eleven girls were studied on Jan. 16–17 and April 9–10; 12 girls on all other dates. All girls in the 15, 16 and 17 year old groups had reached puberty; none of the girls studied on April 9–10 and March 12–13 had reached puberty; 5 girls on Jan. 23–24, 2 on March 19–20, 4 on Feb. 13–14, and 11 on March 5–6 had *not* reached puberty.

[2] Using a respiratory quotient of 0.79 and 3.086 calories as the calorific equivalent of a gram of carbon dioxide.

The striking influence of age is shown by the values for the heat production per kilogram of body weight, which indicate a specific high metabolism with the younger ages. These values for the heat per kilogram have been plotted, referred to age, in figure 1, and a line representing the general trend of the metabolism has been laid on the chart. From this chart we have derived the predicted values for the heat per kilogram of body weight of girls, depending upon age, and these values we present in table 2 as representing the best available standard for predicting the heat production of young girls of this age-range.

TABLE 2

Basal Heat Production per Kilogram per 24 Hours Predicted from Age, for Girls from 12 to 17 Years of Age

AGE	PREDICTED HEAT PER KILO. PER 24 HOURS
Yrs.	Cals.
12	30.9
$12^1/_2$	29.9
13	28.8
$13^1/_2$	27.7
14	26.7
$14^1/_2$	25.7
15	24.6
$15^1/_2$	23.6
16	22.6
$16^1/_2$	21.7
17	21.2

Other general conclusions derived from the study are the following: The average, minimum, resting pulse rate per minute of girls from 12 to 17 years of age, just before rising in the morning, was found to be 81 at 12 years, 77 at 13 years, 77 at 14 years, 83 at 15 years, 71 at 16 years, and 74

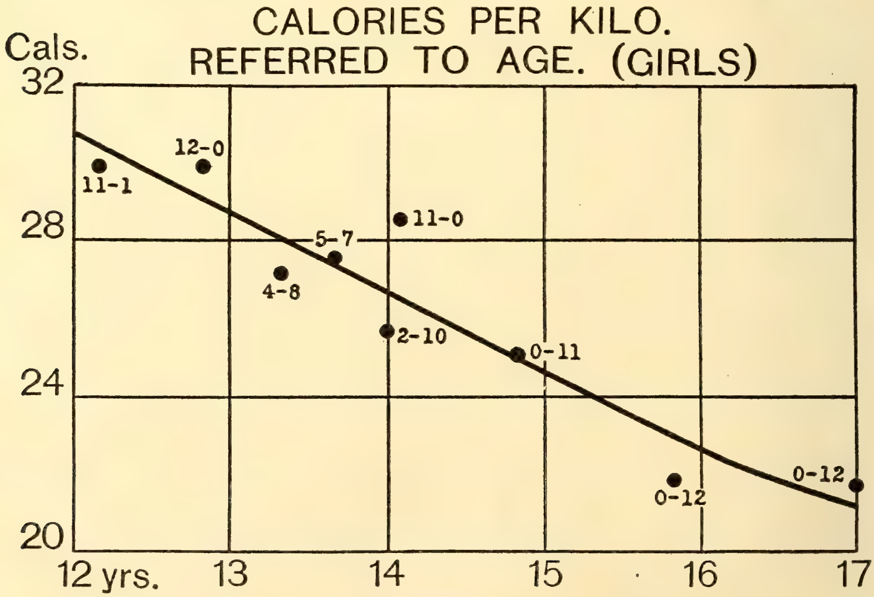

FIGURE 1

Basal heat production of girls per kilogram per 24 hours referred to age

The first of the two figures accompanying each point on this chart represents the number of girls in the group who had *not* as yet menstruated; the second figure indicates the number who had.

at 17 years. The insensible perspiration for these girls per kilogram of body weight per hour was as follows: 0.72 gram at 13 years, 0.71 gram at 14 years, and 0.77 gram at 15 years. The respiratory quotients of groups of 12 girls each, about 7 to 8 hours after a light meal, were 0.81, 0.81, 0.78, and 0.79.

The basal heat production per kilogram of body weight per 24 hours decreases regularly with increasing age from 29.9 calories at 12 years 2 months to 21.7 calories at 17 years. The heat production per square meter of body surface per 24 hours likewise decreases, but not so regularly, with increasing age, ranging from 928 calories at 14 years to 745 calories at 16 years.

The metabolism of groups of young girls can be predicted from the general curve indicating the heat production per kilogram of body weight referred to age to within an average error of ±3.1%. The prediction of the heat production per unit of body weight is somewhat better than that per unit of surface area. The curves representing the heat production per kilogram of body weight referred to weight and per square meter of body surface referred to weight for these groups of girls from 12 to 17 years of age blend with remarkable uniformity with similar curves based upon the measurement of a large number of normal girls from birth to 12 years of age.

No influence of puberty or the prepubescent stage is clearly proven in any of the results.

The details of the entire research are shortly to appear in the *Boston Medical and Surgical Journal.*

ON THE REFLECTION AND RE-EMISSION OF ELECTRONS FROM METAL SURFACES: AND A METHOD OF MEASURING THE IONIZING POTENTIAL OF SUCH SURFACES

By R. A. Millikan and I. G. Barber

Ryerson Physical Laboratory, University of Chicago

Read before the Academy, November 16, 1920

In making photo-electric measurements upon suspended droplets of mercury, one of us has repeatedly found that photo-electrons are always released when the droplet is illuminated by the ultraviolet line of wave-length 2536 Angstroms. On the other hand, it has been frequently proved that the molecules of mercury vapor are not ionized at all by these wave-lengths. The quantum of energy, then, which must be incident upon mercury in the liquid form in order to detach electrons from it is quite different from that required to detach electrons from the free atoms. It is to be expected from this fact that the "ionizing potential" also of liquid mercury will be quite different from that of the same element in the gaseous form. A considerable group of investigations has accordingly been begun in the Ryerson Laboratory for the purpose of determining, in the best vacua obtainable with modern means, the effect of surface conditions upon the emission of electrons from solids and liquids. Preliminary results from one of these investigations are reported herewith.

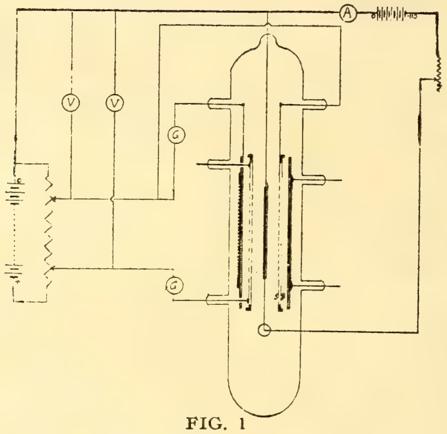

FIG. 1

The aims sought were:

1. To differentiate between the primary electrons which are reflected from a copper surface, and those which are emitted by such a surface because of the bombardment of the primaries.

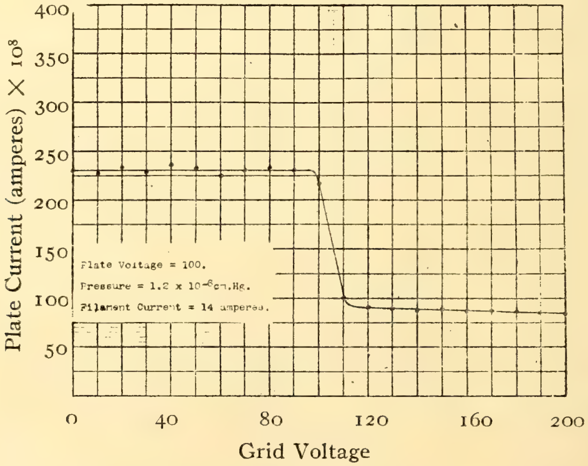

FIG. 2

2. To determine the exact potential at which the surface begins to re-emit under such bombardment. This will be called the ionizing potential of the surface.

3. To determine the number of electrons re-emitted per impinging electron as a function of the energy of impact.

4. To remove surface films by heat and to determine the effect of such films both upon the critical potential (2) and upon the amount of re-emission (3).

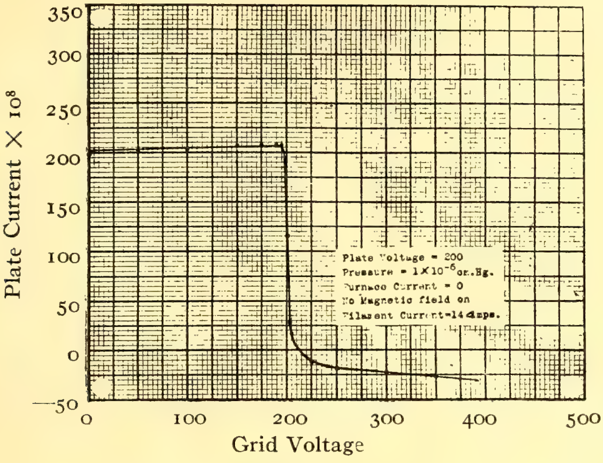

FIG. 3

5. To determine the effect of temperature alone upon the critical potential and the amount of re-emission after the surface had been well cleaned by the method of (4).

The apparatus designed in the endeavor to realize these aims and with which the following results have been obtained is shown, reduced to one-fourth natural size, in figure 1.

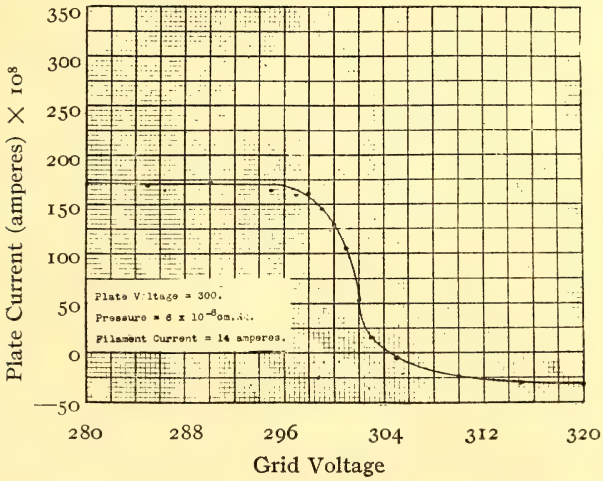

FIG. 4

The thermionic emittor consists of a central platinum cylinder 3 mm. in diameter, coated with the oxides of barium and calcium, and heated indirectly by a central tungsten wire (No. 22) from which it is separated by alundum cement. The emittor is thus *an equipotential surface.* Concentric with this emitting surface is a copper cylinder, *s*, 1.8 cm. in diameter, which will be called grid No. 1. About one-fiftieth part of its surface is perforated with holes, uniformly placed, 1 mm. in diameter. Just outside this cylinder and insulated from it by quartz or porcelain rings 1 mm. thick is a second clyindrical grid, *g*, the holes of which are slightly larger (3 mm. in diameter) but exactly match those of grid No. 1. Outside this is a third cylinder of copper, 3 cm. in diameter, which will be called the plate. The plate can be heated by means of a coil of nichrome wire wrapped about it but insulated from it with a soap-stone coating 2 mm. thick.

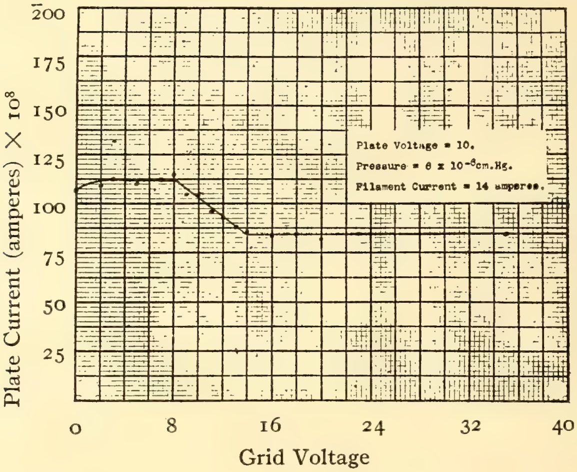

FIG. 5

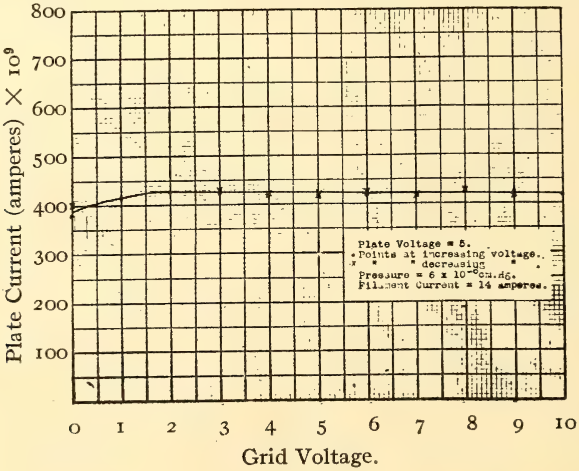

FIG. 6.

In the following observations the plate and grid No. 1 are kept at a common potential and both at a fixed potential, e. g., 100 volts, above the emittor. Figure 2 shows the curve obtained when the plate-current is plotted against the potential of grid No. 2 as the potential of the latter is varied from 0, the potential of the filament, up to 200 volts. When the potential of grid No. 2 has risen above that of grid No. 1, that is, above 100 volts, the secondary electrons, emitted by the plate under the bombardment of the primary electrons having an energy of 100 volts, are drawn from the plate to grid No. 2 and thus reduce the current flowing to the plate. Further, the difference between the two constant currents above and below the 100 volt value of abscissae is a measure of the secondary electrons which are attracted back from the plate. The ratio of this difference to the initial constant plate-current gives the number of secondary electrons produced at the plate per primary electron.

Again the difference between 100 volts and the potential at which the current begins to descend (see fig. 2) gives the maximum velocity of emission of secondaries in volts. Figures 3 and 4 show similar curves taken with primary electrons having energies of 200 and 300 volts, respectively, the latter being made with largely expanded abscissae so as to permit of a study of the distribution of velocities of the secondary electrons. It will be seen from figure 4 that practically no secondaries have a velocity of more than 5 volts, even when the exciting primary electrons have velocities of 300 volts. Further, figure 4 shows, since the potential reaches 310 volts before the current is again constant, that about 10 volts are required to pull out of the plate all of the secondaries generated. This is doubtless due to the fact that some of them are shielded by, or lost in, the pits and hollows in the surface of the plate.

Figures 5 and 6 show the curves obtained with primary electrons having energies of 10 and 5 volts, respectively. It will be seen that the maximum velocity of the secondaries obtainable from figure 5 is 2 volts. *The failure of the curve of figure 6 to drop at all as the potential of grid No. 2 passes through 5 volts seems to show conclusively that none of the 5 volt incident electrons are reflected, and also that the ionizing potential of the surface is above 5 volts; while figure 5 shows that it is below 10 volts.*

The accompanying table shows how the coefficient of secondary emission depends upon the energy of impact of the primary electrons in the case of a copper surface which had been subjected to red-hot temperatures for many hours. The coefficients were some 30% less before such treatment. On the other hand, increase in the temperature of the plate after treatment appeared to decrease the coefficient.

TABLE I

Impact voltage	5	7	10	15	20	25	50	75	100	150	200	250	300	400	500
Coef. of secondary emission	0	.08	.22	.30	.34	.40	.46	.58	.65	.86	1.00	1.10	1.26	1.25	1.25

The important results so far obtained from this work, which is still in progress, may be summarized as follows:

1. Contrary to current belief,[1] there appears to be no such phenomenon as the *direct reflection of* an electron from a copper surface, or if the phenomenon exists at all, the number so reflected at potentials less than the ionizing potential is negligible.

2. The result obtained in (1) reveals, therefore, a method of determining with simplicity and with considerable accuracy the ionizing potential of a metal surface. (The careful determination of the values of such potentials is now being carried out.)

3. As the energy of electronic impact increases, the amount of reemission from copper increases up to about 300 volts, from which point on it remains essentially constant.

4. In the case of copper, the number of electrons detached per impinging electron is never found to exceed 1.3.

5. The number of secondaries per electron is somewhat increased by cleaning the surface by heat, and it appears to be somewhat decreased when the temperature of the surface is raised.

6. The maximum energy of emission of electrons released by electronic bombardment of copper increases from about 2 volts to about 5 volts as the energy of impact increases from 10 volts to 300 volts.

The full details of this work will be reported by one of us in a more extended article in the *Physical Review*.

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[1] von Baeyer, O., *Verh. deut. physik. Ges., Berlin*, **10**, 1908 (903); also Gehrts, A., *Ann. Physik., Leipzig*, **36**, 1911 (996). Horton and Davies, *Proc. Roy. Soc.*, **97**, 23, 1920, obtained results which they interpret in terms of reflections, but which we do not think demand such interpretation. The article by Hull, A. W., *Proc. Inst. Radio Engineers*, Feb., 1918, should also be read, though it does not bear upon this particular point.

CHRONOLOGY OF THE SAN JUAN AREA

By Earl H. Morris

American Museum of Natural History, New York

Communicated by H. F. Osborn, December 2, 1920

The chronological outline herein presented is the result of ten years' research among prehistoric aboriginal remains in that portion of the San Juan watershed lying in northwestern New Mexico, southwestern Colorado, and adjacent areas in Arizona and Utah. In the earlier stages of the work, financial support was given successively by the School of American Archaeology, the St. Louis branch of the Archaeological Institute of America, and the University of Colorado. For the past five years the work has been under the direction of the American Museum of Natural History as a part of the Archer M. Huntington Survey of the Southwest.

Approximately 250 archaeological sites have been visited and examined, 148 of them intensively. The distribution of the letter is as follows:

Immediate valley of the San Juan	22
East and south side of river:	
Canyon Frances	3
Canyon Gobernador	12
Canyon Largo	1
Kootz Canyon	1
Chaco Valley	22
Eastern slope, Tunicha Mountain	20
North side of river:	
Animas Valley	27
La Plata Valley	14
Between La Plata and Mancos rivers	26

For the present, the San Juan culture area may be considered coextensive with the watershed of the river whence it takes its name. This area is some 25,000 square miles in extent. It is a vast elevated plateau cut through by a few perpetual streams and numerous cliff-bordered washes which flow only during spring thaws or subsequent to heavy rains. Despite its general aspect of intense aridity, aboriginal culture flourished over the entire area to the extent that almost every locality capable of supporting an agricultural people is marked by ruins large or small. These remains belong to four major cultural horizons, in stratigraphic order as follows:

IV. Late Black-on-white
III. Early Black-on-white
II. Pre-Pueblo
I. Basket Maker

1. *Basket Maker Period.*—The chief diagnostic characters of Basket Maker culture are: Pronounced dolichocephalic crania, without artificial occipital flattening; apparent total absence of permanent habitations; absence of bow and arrow; use of *atlatl;* high development of textile manufacture, with specialized types of sandals and burial baskets; absence of pottery; and the cultivation of one very primitive type of corn.

Remains of the Basket Maker period have been found only in a restricted area, all identified sites being within a radius of twenty-five miles of Bluff City, Utah. This region has been intensively worked by Dr. A. V. Kidder and Mr. Samuel H. Guernsey. Hence, to them belongs the credit for having identified and characterized this initial culture stratum.

2. *Pre-Pueblo Period.*—Crania from pre-Pueblo sites are for the most part dolichocephalic, like those of the Basket Makers; but among them there are a minority of what appear to be brachycephalic crania with artificial occipital flattening, such as is typical of the subsequent periods.

Pre-Pueblo dwellings were flimsy, irregularly arranged, one-storied structures, with walls usually composed of poles set upright and plastered with mud. Occasionally, however, the basal portions of walls were faced with stone slabs set on edge, or more rarely, a wall was raised of crude masonry. Within the area of or adjacent to the buildings were circular subterranean or semi-subterranean chambers whose walls were formed by the sides of the excavated pits. These chambers were the prototype of the later kiva.

Perishable materials have not been recovered from this period. Pottery was made and commonly used, but both forms and ware were crude. Smooth ware predominated over corrugated, which is doubtfully present. On the average less than thirty per cent of the smooth ware bears painted decoration.

Pre-Pueblo remains have been found in every ruin-strewn locality where a search for them has been made. The sites occur singly and in groups, usually in open, unprotected places. As type sites may be mentioned those upon the mesas between the La Plata and Mancos rivers described elsewhere (Thirty-third Annual Report, Bureau of American Ethnology, Washington, 1919).

3. *Early Black-on-white Period.*—Crania from the early black-on-white period are all of apparent brachycephalic type with artificial occipital flattening. Dwellings of this period are more numerous and more widely distributed than those of earlier or later types. Both large and small structures occur, the latter being more common. Stone was the building material generally used, but the individual blocks were never faced. In some cases the buildings may have been more than one story in height. The usual forms are rectangular, L or E-shaped. Circular subterranean chambers possessing all diagnostic kiva characters form a component part of every dwelling.

The textiles of this period are in every sense of Pueblo type. There is no remaining trace of the specialized Basket Maker bags, burial baskets, and square-toed sandals. At least three types of corn were cultivated. Ceramics was highly developed, both smooth and corrugated ware being very abundant. Certain forms of smooth ware, such as pitcher-shaped drinking vessels, and half-gourd ladles, and the absence of exterior decoration on bowls, are characteristic. Corrugated ware reached its highest development, as expressed by wide range of forms and extreme variety of decoration produced by the manipulation of the coils.

No early black-on-white site has been thoroughly excavated; hence, a type site remains to be described.

4. *Late Black-on-white Period.*—Crania from the late black-on-white period are, in superficial appearance, like those from the preceding period; that is, short from back to front with artificial occipital flattening.

Ruins belonging to this period are relatively few in number, but are generally of large size. Typically, they are compact massive structures, two, three, and four or more stories in height, rising terrace fashion from an enclosed court. Stone was the customary building material. Many of the blocks were faced with admirable precision, and some of them were even polished. Kivas occurring as components of these structures are of large size, highly elaborate, and of unsurpassed construction. They are situated both within the enclosed courts and incorporated within the main building masses, the former subterranean, the latter above ground, and in some cases raised.

Specialization in architecture is much in evidence, several distinct local types, each distributed over a continuous minor area, being recognizable. The two prevailing localized forms are the cliff-houses, Mesa Verde as the type, and the great houses of the Aztec-Chaco region, the large ruin at Aztec, New Mexico, now nearing complete excavation by the writer, as the type specimen.

Cloth, basketry, and sandals are in every known respect identical with those of the preceding period. Smooth and corrugated ware continued through late black-on-white time; smooth ware reached the zenith of its perfection, while corrugated ware began to degenerate. Local specialization in pottery was a direct accompaniment of specialization in architecture. There are at least three major local types of pottery, namely, the Kayente, the Mesa Verde, and the Chaco, each with diagnostic forms and styles of ornamentation.

The following is a tentative interpretation of the data given above. In the Basket Maker period there is evidence of a dolichocephalic people, skilled makers of textiles, but ignorant of the utilitarian possibilities of clay and of the use of the bow and arrow. They were essentially nomadic, without permanent dwellings, but undergoing transition to sedentary life under the compelling influence of the cultivation of corn, of which they had a single primitive type.

In the pre-Pueblo period there may be postulated the blending of a brachycephalic element with the previous dolichocephalic stock. In this period a crude architecture was developed, the bow and arrow were introduced, and the manufacture of pottery begun.

In the early black-on-white period the brachycephalic stock seems entirely to have supplanted the dolichocephalic. Diagnostic Pueblo architecture expressed in fairly compact structures consisting of rectangular living rooms and circular ceremonial chambers, had taken definite form. Agriculture had been amplified by the addition of at least two types of corn and pottery making had been developed into a fine art.

In the culminating, or late black-on-white period, there are no evident changes in physical characteristics. Some potent integrating force, perhaps the pressure of predatory nomadic intruders, had brought together the hitherto widely scattered population into a few large and powerful groups. These groups, in their several localities, carried Pueblo culture to its highest material perfection, at which point, for reasons as yet undetermined, Pueblo occupation of the San Juan area came to an abrupt close. As a terminating phase of late black-on-white time, there came a period of redistribution during which Pueblo culture was carried to the east, south and west.

The correlation of specific San Juan chronology with the chronology of the southwest as a whole is a work much of which is still to be done. The only other portion of the southwest for which a definite chronology has been proposed is that for the upper Rio Grande Valley in New Mexico. This area is both historic and prehistoric, in contrast to the area of the San Juan, which is exclusively prehistoric. Thus, the systematic series for the former presents a sequence, part of which can be historically dated, as established by Mr. N. C. Nelson and Dr. A. V. Kidder. The chronology of this Rio Grande area contains the following major divisions:

5. Modern Painted Ware
4. Historic Two-Color Glazed Ware
3. Three-Color Glazed and Painted Ware
2. Two-Color Glazed Ware
1. Two and Three-Color Ware

No. 1 of this series is comparable to IV in the San Juan series. Our data may, therefore, be interpreted as indicating a shift of cultures from the San Juan to the Rio Grande. Since ceramic types are but the horizon markers for the cultural epochs to which they belong, the chronologies for the San Juan and the Rio Grande areas may be combined and re-cast as in the following:

1. Basket Maker, or Initial Period
2. Pre-Pueblo, or Formative Period
3. Early Two-Color Painted Ware, or Period of Development
4. Late Two-Color Painted Ware, or Period of Culmination and Redistribution
5. Two-Color Glazed Ware
6. Three-Color Glazed and Painted Ware
7. Historic Two-Color Glazed Ware

It appears, therefore, that the establishment of a chronological scale for the area of the San Juan gives us a succession of cultural periods during the greater part of the prehistoric period for the southwest.

A full description of the sites examined, together with a formulation of the evidence upon which the preceding chronological sequence rests will be presented in future numbers of the *Anthropological Papers of the American Museum of Natural History.*

SOME GEOLOGIC CONCLUSIONS FROM GEODETIC DATA

By William Bowie

Division of Geodesy, U. S. Coast and Geodetic Survey

Communicated by A. O. Leuschner. Read before the Academy, November 16, 1920

For a number of years geodetic data were collected for the purpose of controlling surveys and maps and for the determination of the shape and size of the earth. But in recent years they are also extensively used in investigations dealing with geological and geophysical problems. These investigations have included data for the United States, India, Canada and a small part of Europe.

The investigations in isostasy have proved that for the areas considered there is approximately the same mass in each column of unit cross-section extending from the surface of the earth down to a depth of, say, 75 miles. This statement is justified regardless of what the depth of compensation actually is, for the deficiency or excess of mass in a few miles at the depth of approximately 75 miles is a small percentage of the mass of the whole column. With regard to depth of compensation it may safely be said that there is no sharply defined surface which is at uniform depth below sea level, limiting the isostatic compensation. It is probable that there is a zone, rather than a surface, which limits the compensation.

We have no evidence as to the manner in which the compensation is distributed vertically, but the uniform distribution, which was adopted in order to make the computations more feasible, is as logical as any other simple method. Regardless of what the method of distribution is actually, the conclusions reached from the isostatic researches cannot be seriously affected.

What is the area of the cross-section of the unit column that may be in equilibrium is a matter which is in doubt, but as data are accumulated the area of the cross-section is successively lessened.

It has been found that large areas are in almost perfect isostatic balance and it seems to be very probable that an area of about 70 miles square is very closely compensated. This is the area of one square degree at the equator, or very nearly so.

It has been found that the distribution of the compensation of a topographic feature horizontally to a distance of about 60 kilometers from the feature gives as consistent results as local compensation, but when the compensation is distributed horizontally to a distance of 170 kilometers the results are not so accordant as with local distribution of the compensation.[1]

It must be held that any system for correcting geodetic data for the effect of topography and isostatic compensation, which makes the computed values of gravity agree closely with the observed values in different

Effect of Gravity Anomalies of the Compensation for Inner Zones

NUMBER AND NAME OF STATION	ELEVATION H	ISOSTATIC ANOMALY IN DYNES	COR. TO DISCARD COMP. OUT TO AND INCLUDING		ANOMALIES WITH ISOSTATIC COMP. OMITTED TO	
			Zone L. (17.9 mi.) In dynes	Zone M. (36.5 mi.) In dynes	Zone L. In dynes	Zone M. In dynes
	Meters					
41 Wallace, Kans.	1005	−0.012	−0.027	−0.048	−0.039	−0.060
42 Colorado Springs, Colo.	1841	−0.007	—0.054	−0.094	−0.061	−0.101
43 Pikes Peak, Colo.	4293	+0.021	−0.070	−0.113	−0.049	−0.092
44 Denver, Colo.	1638	−0.016	−0.038	−0.076	−0.054	−0.092
45 Gunnison, Colo.	2340	+0.020	−0.063	−0.120	−0.043	−0.100
46 Grand Junc., Colo.	1398	+0.024	−0.041	−0.082	−0.017	−0.058
47 Green River, Utah	1243	−0.021	−0.033	−0.067	−0.054	−0.058
48 Pleasant Val. Junc., Utah	2191	+0.004	−0.060	−0.103	−0.056	−0.099
49 Salt Lake City, Utah	1322	+0.010	−0.040	−0.075	−0.030	−0.065
50 Grand Canyon, Wyo.	2386	−0.002	−0.061	−0.108	−0.063	−0.110
51 Norris Geyser Basin, Wyo.	2276	+0.021	−0.059	−0.104	−0.038	−0.083
52 Lower Geyser Basin, Wyo.	2200	−0.001	−0.058	−0.103	−0.059	−0.104
55 Mt. Hamilton, Cal.	1282	−0.003	−0.014	−0.017	−0.017	−0.020
63 El Paso, Tex.	1146	+0.007	−0.030	−0.054	−0.023	−0.047
64 Nogales, Ariz.	1181	−0.050	−0.029	−0.046	−0.079	−0.096
67 Goldfield, Nev.	1716	−0.013	−0.043	−0.074	−0.056	−0.087
68 Yavapai, Ariz.	2179	+0.001	−0.045	−0.080	−0.044	−0.079
70 Gallup, N. Mex.	1990	−0.013	−0.053	−0.095	−0.066	−0.108
71 Las Vegas, N. Mex.	1960	+0.003	−0.053	−0.094	−0.050	−0.091
75 Lead, S. Dak.	1590	+0.052	−0.038	−0.064	+0.014	−0.012
81 Sisson, Cal.	1048	−0.010	−0.033	−0.058	−0.043	−0.068
82 Rock Springs, Wyo.	1910	+0.013	−0.052	−0.093	−0.039	−0.080
98 Alpine, Tex.	1359	+0.021	−0.034	−0.061	−0.013	−0.040
99 Farwell, Tex.	1259	−0.016	−0.030	−0.055	−0.046	−0.071
102 Cloudland, Tenn.	1890	+0.004	−0.025	−0.039	−0.021	−0.035
109 Sheridan, Wyo.	1150	+0.032	−0.032	−0.068	0.000	−0.036
110 Boulder, Mont.	1493	−0.015	−0.046	−0.077	−0.061	−0.092
114 Truckee, Cal.	1805	−0.028	−0.051	−0.085	−0.079	−0.113
115 Winnemucca, Nev.	1311	−0.009	−0.032	−0.062	−0.041	−0.071
116 Ely, Nev.	1962	−0.021	−0.055	−0.094	−0.076	−0.115
117 Guernsey, Wyo.	1322	+0.036	−0.031	−0.062	+0.005	−0.026
195 Lander, Wyo.	1635	+0.019	−0.047	−0.090	−0.028	−0.071
198 Edgemont, S. Dak.	1066	+0.054	−0.028	−0.052	+0.026	+0.002
202 Moorecroft, Wyo.	1295	+0.021	−0.031	−0.058	−0.010	−0.037
269 Hill City, S. Dak.	1518	+0.042	−0.040	−0.067	+0.002	−0.025
270 Newcastle, Wyo.	1328	+0.029	−0.035	−0.064	−0.006	−0.035
271 Bridgeport, Neb.	1114	−0.008	−0.029	−0.053	−0.037	−0.061
272 Buford, Wyo.	2396	+0.046	−0.057	−0.100	−0.011	−0.054
273 Boulder, Colo.	1630	−0.014	−0.048	−0.092	−0.062	−0.106
274 Lafayette, Colo.	1595	−0.020	−0.040	−0.081	−0.060	−0.101
275 Brighton, Colo.	1511	−0.006	−0.038	−0.073	−0.044	−0.079
276 Idaho Springs, Colo.	2303	+0.022	−0.068	−0.120	−0.046	−0.098
Mean with regard to sign		+0.005	0.043	0.077	−0.037	−0.072
Mean without regard to sign		0.019			0.040	0.072

parts of the earth and for various types of topography, is very close to the ideal or true system. While the isostatic investigations have been confined to only a portion of the earth's surface it seems probable that all land areas will be found to be in isostatic equilibrium.

A test was made to show whether the compensation of small areas of topography could be ignored without seriously affecting the results. Forty-two stations having high elevations were selected for the test. The elevations of these stations varied from 3000 to 14,000 feet. When the compensation of the topography for a circle with a radius of 17.9 miles was ignored the gravity anomalies became much larger and 37 of the 42 anomalies had negative signs. When the area of the circle was given a radius of 36.5 miles the anomalies were still further increased and all of the anomalies except one had the negative sign. With the small circle the mean anomaly with regard to sign is −0.037 dyne while with the larger circle the mean anomaly with regard to sign is −0.072 dyne. When it is considered that the mean anomaly with regard to sign for these 42 stations under consideration, all at high elevations, is only +0.005 dyne, it is realized that the ignoring of the local compensation is not justified. The table preceding shows these results.

This test shows that, for even very small areas, the topography is at least largely compensated. This is a most important conclusion for by analogy no such mass as the sedimentary material forming the delta of a large river entirely escapes isostatic compensation as has been held by some investigators.[2]

There are very definite relations between the gravity anomalies and the Cenozoic and Pre-Cambrian formations. The Cenozoic stations have negative anomalies in nearly all cases, and the largest anomalies are found at stations located on this formation. The anomalies at the stations on the Pre-Cambrian areas of limited extent are nearly all positive.[3]

The writer made an investigation of these relations and arrived at the conclusion that the negative sign of the Cenozoic anomalies is due in great part to the abnormally light material of that formation. Of course it is necessary to consider the effect of the Cenozoic material which extends below sea level. Nearly all Cenozoic formations are at comparatively small elevations and the abnormal density of the material which may be above sea level could account for only a very small part of the anomalies at Cenozoic stations.

In a similar manner it was found that the existence of abnormally dense material in the Pre-Cambrian formation will account for the positive sign of the anomalies at stations on that formation. It will be necessary, as in the case of the Cenozoic formation, to consider the material that extends below sea level.

These are logical conclusions for it is not conceivable that with a whole country (the United States, for example) in practically perfect isostatic balance, the column under an area of sedimentation should be too light and that the column under an area subjected to long erosion should be too heavy.

The writer's conclusion has been confirmed by a recent investigation in India.[4]

If the columns under the Cenozoic and the Pre-Cambrian formations are in isostatic equilibrium, then there must exist the compensation not only of the material which is above sea level but of the deficiency of matter in the Cenozoic and the excess of matter in the Pre-Cambrian material which exist below sea level.

It is inconceivable that 20,000 or 30,000 feet of sediment all deposited at a low elevation, approximately sea level, could take the place of an equal volume of material presumably of normal density and still have the column contain normal mass, without there having occurred an increase in density in the material of the column to balance the deficiency of mass in the volume occupied by the recent sedimentary material. If the Cenozoic material is 30,000 feet thick, the deficiency in mass is about 3000 feet, and the isostatic compensation will be equal to this amount. In the 75-mile column the increase in density necessary to balance the Cenozoic deficiency will be about $^3/_4$ of 1%, if we assume that the average density throughout the column is approximately three. If the density is greater than that, the percentage will be proportionately increased.

In order to maintain the isostatic balance a mass equivalent to 27,000 feet of Cenozoic material must have been transferred from the column under the Cenozoic formation.

The contraction of the material in the column under the Cenozoic formation may have begun before sedimentation was initiated, but further contraction and consequent increase in density must have taken place with the sedimentation for, otherwise, the surface of the column would have stood at some times much above sea level, which seems to have been improbable.

The evidence *available from geodetic investigations* indicates in the strongest way that land masses are in equilibrium and that this equilibrium exists in comparatively small areas. A natural inference is that land masses have been equally in equilibrium in former geologic periods.

How, then, can an area of sedimentation at approximately sea level, of one age, be a mountainous area in a succeeding one?

Mountain formation by transportation of material horizontally from one column in equilibrium to another column in equilibrium could not take place without destroying the isostatic balance. Of course, material is transported from one column to another (surface transportation after erosion is not included in this statement) but this is when the columns are out of balance and the movement renews the isostatic equilibrium between the two columns. Mountain masses are not excess loads on the earth, as is proved by the existence of the isostatic conditions in mountainous regions, therefore the materials forming the mountains were not moved horizontally to the region. We are forced to the conclusion that the mountains must result from vertical movements in the columns under them. The vertical movement evidently must be due to an expansion and consequent decrease in density in the material of the column.

When we consider that all extensive areas of recent sedimentation on which we have gravity stations are in isostatic equilibrium and that mountain systems formed in previous sedimentary areas are also in equilibrium, we have no alternative to the view that an actual expansion of the columns under the mountains has taken place.

The objection will be raised that there is abundant evidence that there have been horizontal movements in the materials forming the mountains. This is granted but is it necessary to go far beyond the mountain area for the forces acting laterally which cause the observed horizontal movements? Can we not conceive that, in the uplifting of the mountains of the Appalachian and Himalayan systems, for instance, the vertically acting forces will cause the material to progress in the directions of least resistance and that these directions may be horizontal in some cases and in others at varying angles with the vertical.

During the period of sedimentation, material probably was not laid down in smooth concentric sheets but in irregular ways. In the process of uplift, due to expansion, the rate of expansion would undoubtedly be different in different parts of the zone. The sediments are of varying thickness, the material on which the sediment had been deposited varies in composition and consequently in its resistance to uplift, and finally there is cubical expansion of the material which must cause the material at the borders, but outside of the columns, to modify somewhat the upward movements of the expanding mass. At least this effect must be exerted near the upper part of the expanding column. The areas affected are not small for the area of the base of the Appalachian system is more than 1000 miles in length and is approximately 200 miles in width, on an average. Consequently, there seems to be sufficient space for the development of lateral movements within the area due to local causes.

The drowned valleys along the continental coasts show that decided subsidences have taken place. How could these have been caused except by contraction of the columns under the affected areas since the pendulum proves rather conclusively that the regions along the coasts are in isostatic equilibrium? Here the reverse process to that of mountain forming operated.

The writer has arrived at the conclusions here presented after endeavoring to harmonize geodetic data and certain observed geological facts. He makes no attempt to formulate a theory as to what agencies are at work to change the density of material far below sea level (but within, say, 75 miles of the surface), but he suggests that the vertical movements accompanying the isostatic adjustments may be a partial cause. Material is moved in this process, to zones of decidedly different temperatures, to hotter or to colder zones, and it seems to be logical to conclude that some chemical or physical changes may take place which would affect the density of the material transported. In the process of isostatic adjustment all the material of a column above the zone of flow is raised or lowered. The ordinary thermal expansion of the material of a column, as it changes its temperature, is capable of accounting for only a small part of the changes in the length of a column.

It should be clearly borne in mind that the theory of isostasy does not explain any vertical movements other than those necessary to maintain equilibrium. Some other theory is needed to explain elevation and subsidence and the writer feels that the theory of local expansion and contraction is in general in harmony with the geodetic data. He believes, however, that there are local vertical movements of small amounts which may not be due to these causes.

It is hoped that increased activity will take place in collecting geodetic data and in extending the investigations in isostasy. It is particularly desirable that we have the investigations include the ocean areas as soon as geodetic data may be available within them.

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[1] See page 90, "Investigations of Gravity and Isostasy," *Spec. Pub. No.* **40,** *U. S. Coast and Geodetic Survey,* 1917.

[2] "Strength of the Earth's Crust," by Jos. Barrell, *J. Geol.,* January-February, 1914 (48); and "Discoidal Structure of the Lithosphere" by Bailey Willis, *Bull. Geol. Soc. Amer.,* June 30, 1920.

[3] "Investigations of Gravity and Isostasy," *Spec. Pub.* **40,** *U. S. Coast and Geodetic Survey,* pp. 70–82.

[4] "Investigations of Isostasy in Himalayan and Neighboring Regions," *Professional Paper No.* **17,** by S. G. Burrard, former Surveyor-General of India, 1918.

EXPERIMENTS ON THE ELECTRICAL CONDUCTION OF A HYDROGEN ALLOY

BY DONALD P. SMITH

CHEMICAL LABORATORIES, PRINCETON UNIVERSITY

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In studies which were described some time since[1] it was found that hydrogen, when discharged electrolytically upon any one of certain metals which occlude it, produces a temporary diminution in the electrical resistance of the metal. With metals, such as palladium, which occlude large amounts, and suffer, as has long been known, a lasting *increase* of resistance, the diminution of resistance, or supplementary conductance, is superimposed upon the opposite and more enduring effect; while in the case of platinum, which has only a small occlusive capacity, and does not display the permanent increase, the supplementary conductance may be observed alone.

Regarding the nature of this peculiar electrical conduction, three explanations seem thus far to offer:

(1) The conduction is by a transient form of the occluded hydrogen, probably monatomic, and consists in a transport of electrical charges between points of different potential within the metal.

(2) The conduction is by unstable hydrides, which are produced only under conditions of electrolysis which are equivalent to a high pressure of hydrogen, and which rapidly decompose when these conditions cease to obtain. (Newbery.)[2]

(3) The conduction, while not the result of chemical combination, is due to a temporary effect which the occluded hydrogen exerts upon the distribution or mobility of the electrons within the metal.

Upon the second or third of these suppositions, there appears to be no reason to expect the character of the supplementary conduction to differ from that of metallic conduction in general; but the first-named hypothesis leads to a different expectation.

This first explanation, which has been suggested by the observation that the supplementary conduction and the volume of the alloy are directly related, may be stated somewhat more in detail as follows:

It is supposed that within a hydrogen alloy there exist (*a*) molecular hydrogen, (*b*) atomic hydrogen, and (*c*) hydrogen combined with the metal to form solid solution; and that these three forms are in mobile equilibrium in some such way as is indicated by the scheme

$$2H + 2xM \rightleftarrows 2MxH \rightleftarrows H_2 + 2xM. \qquad (1)$$

Since the fact is well established that the formation of solid solutions between two metals brings about a great increase in the electrical resistance beyond the value which would result from simple mixture, it is natural to attribute the permanent increase of resistance in hydrogen alloys such as palladium-hydrogen to this combination; and if the supplementary conduction is attributed to the atomic hydrogen, it is evident that, upon the conception just outlined, this conduction must vary with the cathodic current density during electrolysis, as has been found to be the case, and must persist after the interruption of electrolysis until the equilibria of equation (1) have become established.*

If the foregoing explanation be essentially correct, the conduction in question is that of a gas confined within the cavities of the metal. It is accordingly not to be expected that it will conform to Ohm's law, but rather that the resistance observed will depend upon the intensity of the current employed in its measurement, or in other words, upon the applied e. m. f.

The experiments about to be described were, therefore, undertaken in order to test this prediction in the case of palladium-hydrogen since the earlier observations, having been made by null methods, gave only the resistance for "zero current."

Experiments.—The cathode investigated consisted of a wire of commercially pure palladium, 0.05 mm. in diameter, and approximately 110 mm. in length. The cell and thermostat were those which have been described elsewhere,[3] as were also the manner of attaching and supporting the cathode wire, save that arrangements for observing changes of length were dispensed with. The electrolyte was 2-*n* sulphuric acid, which was caused to circulate during the experiment, as heretofore. The temperature was maintained throughout at 25.00 ± 0.02° C.

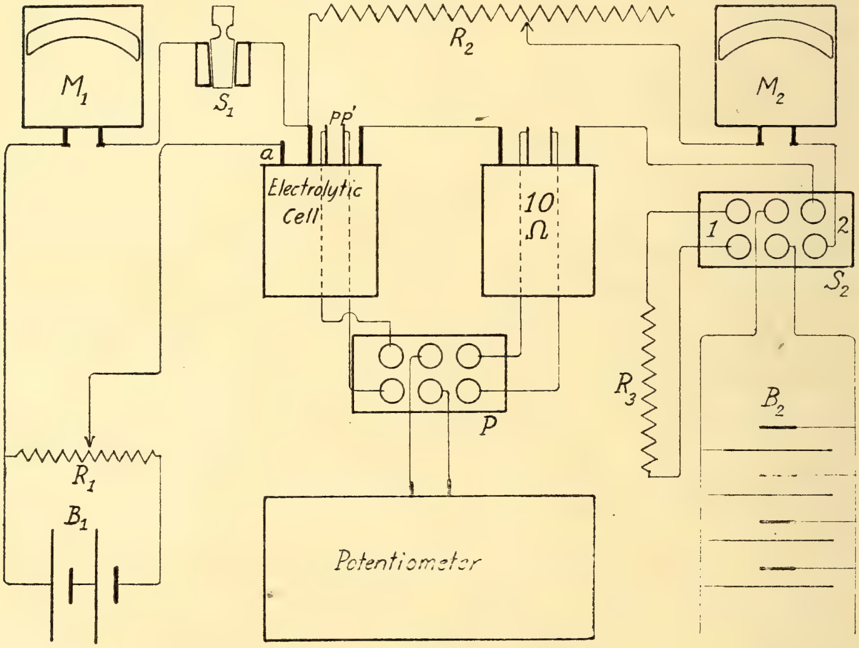

FIG. 1

Electrical connections

The electrical connections are represented diagrammatically in figure 1, where the electrolytic circuit is shown at the left (subscripts "1"); and the resistance-measuring circuit at the right (subscripts "2").

An experiment was conducted as follows: The switch S_2 being in position 1, so that the battery B_2 was short-circuited through R_3 and the resistance-measuring circuit was open, the electrolytic circuit was closed by means of the plug switch S_1, and was regulated with the variable resistance R_1 until the millammeter M_1 showed the current corresponding to the desired cathodic current density. Electrolysis was then continued, with occasional regulation of the current, until the cathode wire had attained saturation, the time required being calculable from the data previously obtained.

After the electrolytic current had been interrupted by withdrawing the plug S_1 the resistance-measuring circuit was closed by throwing the switch S_2 into position 2. The current, derived from the battery B_2, which consisted of four large storage cells in parallel, was now regulated by means of the resistance box R_2 until the millammeter M_2 gave the reading sought. This was from 2 to 4 milliamperes, these values being selected upon the consideration that for the sake of constancy the current should be as small as the precision of the potentiometer would permit.

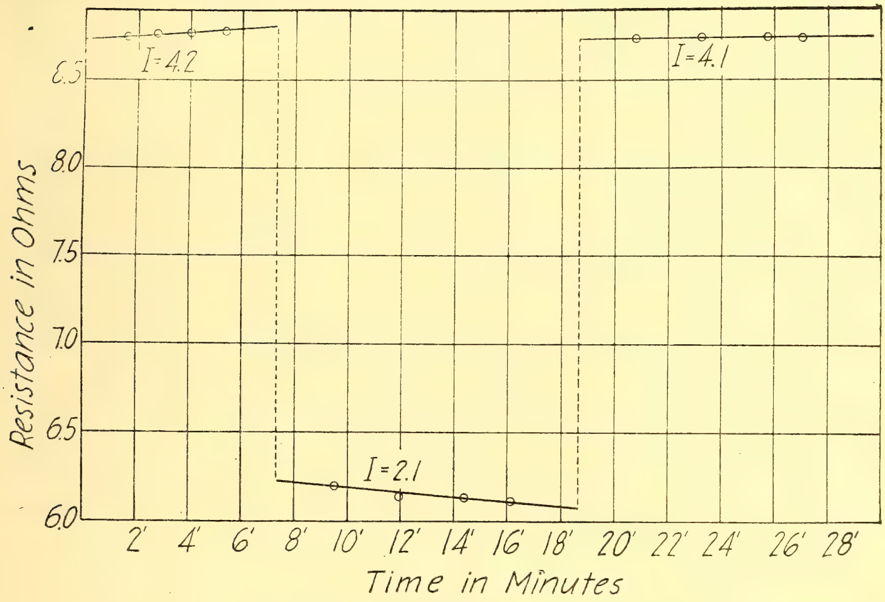

FIG. 2

Dependence of resistance on current

By means of the Pohl commutator P the potentiometer was now connected alternately to the potential leads, p and p', coming from the ends of the cathode wire, and to those of the ten-ohm comparison resistance, readings of the fall of potential across the two resistances being taken several times in rapid succession, while the time was in each case noted, in order that the value for the ten-ohm might be calculated by interpolation for the exact moment at which that for the cathode wire was observed. This made it possible to eliminate the error which would otherwise have resulted from the gradual drift in the resistance of the cathode, and hence in the measuring current.

A complete experiment consisted of three such series of observations as that which has just been described, obtained at intervals as short as possible. The first and last were taken with a measuring current of approximately 4 milliamperes, and the intermediate one with a current of half this intensity.

The potentiometer employed was a low-resistance "Type K" instrument by Leeds and Northrup, with which was used a suspended coil galvanometer by the same makers. The latter was adjusted by means of a shunt until its vibrations were aperiodic, and had under these conditions a sensibility more than adequate. Care was taken before the beginning of an experiment to bring the potentiometer battery to such constancy of e. m. f. that errors from this source were negligible, and this constancy was checked at least once in the course of each series of observations by comparisons with a set of standard cells.

The ten-ohm comparison resistance used was not a precision coil, so that the resistances found are only relative. Their relative accuracy may be estimated at 0.02% in the least favorable instances.

Results.—The results of one experiment are recorded in table I, and displayed graphically in figure 2. Before this run the wire had been several times saturated with hydrogen, and immediately before the interruption of electrolysis the current had been maintained over night at a value of 2.3 milliamperes, corresponding to a cathode current density of about 1.4 amperes per square decimeter.

In the table below, column 1 shows the measuring current in milliamperes; column 2 the time elapsed from the first reading in seconds; and column 3 the interval between successive observations. The unenclosed numbers in the column headed E_{10} give the observed fall of potential across the ten-ohm resistance, and the numbers in parentheses are interpolated from the preceding for the time at which E_x was read. E_x is the observed fall of potential across the cathode wire; and R_x is the resistance of the latter calculated from the relation $R_x = 10 \cdot E_x/(E_{10})$.

In figure 2 the values of t from table I are plotted as abscissae, and those of R_x as ordinates. If the resistance were independent of the measuring current, the values for all three series of the table should lie upon an unbroken curve which is very nearly rectilinear; but it will be seen that the points for the intermediate series, taken with a measuring current only half as great as that used in the two end series, fall far out of line. A diminution of current of 50% has produced a diminution of resistance of some 30%.

This result was confirmed by that of an entirely similar experiment in which the changes of current and of resistance were, respectively, fifty and forty-two per cent; and finally by that of an experiment in which the electrolyte was withdrawn from the cell, before the resistance of the wire was observed, and in which the corresponding decrements were fifty and twenty-six per cent. It should be pointed out that only qualitative agreement between the several experiments was to be expected, since the magnitude of the effect is dependent both upon the quantity and

TABLE 1

i	t	Δt	E_{16}	E_x	R_x
4.2	0		0.039556		
		98			
	98		(0.039561)	0.034600	8.7459
		34			
	132		0.039564		
		33			
	165		(0.039561)	0.034663	8.7619
		33			
	198		0.039558		
		43			
	241		(0.039558)	0.034695	8.7656
		41			
	282		0.039559		
		38			
	320		(0.039557)	0.034733	8.7805
		38			
	358		0.039555		
2.1	467		0.020035		
		104			
	571		(0.020217)	0.012521	6.1933
		61			
	632		0.020338		
		87			
	719		(0.020338)	0.012488	6.1402
		35			
	754		0.020338		
		110			
	864		(0.020338)	0.012479	6.1358
		43			
	907		0.020338		
		61			
	968		(0.020338)	0.012435	6.1142
		30			
	998		0.020338		
4.1	1157		0.039604		
		91			
	1248		(0.039540)	0.034522	8.7334
		59			
	1307		0.039563		
		88			
	1395		(0.039564)	0.034583	8.7410
		55			
	1450		0.039564		
		93			
	1543		(0.039564)	0.034547	8.7319
		30			
	1573		0.039564		
		48			
	1621		(0.039564)	0.034572	8.7382
		29			
	1650		0.039564		

upon the condition of the occluded hydrogen at the time of observation.

That the results were not materially affected by fluctuations of the measuring current is probably sufficiently attested by the regularity of the observations in each series. The precaution was nevertheless taken of substituting for the palladium wire a nearly equivalent manganin resistance, and conducting with this a similar series of measurements. The resistances found were in this instance constant to the fourth significant figure, showing that with the battery employed, which had a capacity of 240 ampere-hours, and supplied only 4 milliamperes or less, at 2 volts, the variations were wholly negligible.

Conclusion.—The fact seems to be established by the foregoing experiments that the temporary supplementary conduction exhibited by metals during cathodic occlusion of hydrogen, and for some time thereafter, is not of ordinary metallic character. Strictly regarded, this has been shown only in the case of palladium-hydrogen; but the great similarity previously shown[3] to exist in the comportment of palladium, of tantalum and of iron toward hydrogen, and of palladium toward oxygen, makes it probable that the supplementary conduction is of the same character in all cases.

The outcome of the experiments is accordingly in conformity with the prediction based upon the first of the three explanations of the conduction which were enumerated above, and lends support to this conception. Upon either of the other two hypotheses the departure from Ohm's law would be difficult to understand.

Obviously, it is to be expected that with measuring currents much higher than those here employed the supplementary conduction would approach a limiting value, owing to the inability of ordinary diffusion to maintain the concentration of hydrogen atoms in the regions from which these are driven by the current. Above this limit the dependence of resistance on current would not be observed. Unfortunately the likelihood of being able to test this further prediction experimentally seems small; for it may be seen from the magnitude of the change of resistance in the range studied that its disappearance can be looked for only at very large values of the measuring current; and with such currents the heating effects would render doubtful any results which might be obtained.

In concluding, acknowledgment should be made of the excellent assistance rendered in these experiments by Mr. Robert F. Mehl.

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[*] The assumption of the existence of molecular hydrogen within the metal seems to be made necessary by certain facts which are not here considered, and the present considerations will not be affected if the quantity of the molecular form is taken to be extremely small, or even zero.

[1] Smith, D. P., and Martin, F. H., *J. Amer. Chem. Soc.*, **38,** 1916 (2577); Harding, E. A., and Smith, D. P., *Ibid.*, **40,** 1918 (1508). Summarized in *Trans. Amer. Electrochem. Soc.*, **34,** 1918 (177).

[2] Newbery, E., *J. Amer. Chem. Soc.*, **41,** 1919 (1887).

[3] Harding and Smith, *loc. cit. supra.*

THE VASCULAR ANATOMY OF NORMAL AND VARIANT SEEDLINGS OF PHASEOLUS VULGARIS

BY J. ARTHUR HARRIS AND EDMUND W. SINNOTT

STATION FOR EXPERIMENTAL EVOLUTION, CARNEGIE INSTITUTION OF WASHINGTON

Communicated by C. B. Davenport, November 29, 1920

The investigations here summarized comprise a comparative and biometric study of the gross vascular anatomy of normal and variant seedlings of *Phaseolus vulgaris.*

Three morphological types have been considered, (*a*) the normal or *dimerous* seedling with two cotyledons and two primordial leaves, (*b*) the *trimerous* seedling with three cotyledons and three primordial leaves, and (*c*) the *hemitrimerous* seedling in which there are three cotyledons and two primordial leaves.

In normal seedlings, the vascular system of the root is typically tetrarch (with four protoxylem poles), and gives rise in the base of the hypocotyl to four pairs of double bundles which soon form a circle of eight bundles which continue to the cotyledonary node. At this point there is a complex vascular anastomosis. From it two strands are given off to each cotyledon. The remainder of the vascular tissue is reorganized into six strands, each of which typically soon divides into two, the twelve bundles thus formed comprising the vascular system of the epicotyl.

The trimerous seedlings typically possess six root poles instead of four, twelve bundles in the hypocotyl instead of eight, and nine primary epicotyledonary bundles instead of six. The nine primary epicotyledonary bundles do not all divide, however, so that the number of bundles in the central region of the epicotyl is variable ranging in general from fourteen to eighteen.

In both classes of seedlings, but more frequently in the normal type, additional or intercalary bundles appear in the hypocotyl, either *de novo* or as a result of division of the primary strands.

Four main groups of problems as to the vascular topography of these seedling types have been considered biometrically: First, the number of bundles at different levels in the seedling; second, the variability in bundle number; third, the differentiation in internal structure of seedlings which are externally dimerous, trimerous and hemitrimerous; and fourth, the interrelationship of bundle number in different regions of the seedling.

The following table of constants[1] summarizes the facts for number and variability of vascular bundles in various regions of the seedling and in the three types of seedlings.[2]

The constants in this table, and the frequency distributions from which the constants were computed, lead to the following conclusions.

	DIMEROUS SEEDLINGS			TRIMEROUS SEEDLINGS			HEMITRIMEROUS SEEDLINGS		
	Mean	S. D.	C. V.	Mean	S. D.	C. V.	Mean	S. D.	C. V.
Root poles									
Minimum	4.01	0.081	2.03	5.02	0.654	13.02			
Maximum	4.13	0.338	8.18	5.16	0.729	14.12			
Mean	4.05	0.171	4.19	5.09	0.707	13.87			
Primary double bundles									
Minimum	4.02	0.140	3.48	5.81	0.288	4.86	5.21	0.608	10.59
Maximum	4.52	0.666	14.74	5.98	0.581	10.01	5.74	0.750	14.07
Mean	4.19	0.411	9.66	5.91	0.405	6.87	5.49	0.676	12.37
Intercalary bundles									
Minimum	0.07	0.261	105.79	0.09	0.292	156.62	0.28	0.449	115.47
Maximum	0.83	1.024	355.48	0.29	0.686	381.67	0.53	1.148	214.68
Mean	0.49	0.687	182.70	0.19	0.491	274.92	0.44	0.737	163.82
Mid-region of hypocotyl									
Minimum	8.11	0.409	5.04	11.99	0.532	4.42	11.36	1.169	9.94
Maximum	10.62	1.645	17.34	12.29	1.283	10.44	12.32	1.524	12.87
Mean	9.23	1.193	12.67	12.16	0.883	7.24	11.94	1.307	10.96
Mid-region of epicotyl									
Minimum	12.11	0.406	3.35	14.89	1.152	7.74	12.93	1.245	9.07
Maximum	12.36	0.757	6.13	16.10	1.750	10.87	14.84	1.778	12.53
Mean	12.22	0.586	4.79	15.47	1.383	8.92	13.83	1.560	11.29

The modal number of primary double bundles in the region of transition from root to stem structure at the base of the hypocotyl is four in the dimerous and six in the trimerous and hemitrimerous seedling. In the normal seedlings more than four bundles may occur, but in no case have fewer than this number been observed. In the trimerous seedling variation both above and below the mode is found, the numbers ranging from four to eight. On the average the number is from 1.38 to 1.89 bundles higher (or from 30.5 to 47.0% higher) in the trimerous than in the dimerous seedlings.

Intercalary bundles, which are rather uncommon in seedling anatomy in general, occur in from 11 to 46% of the normal seedlings, whereas they are found in only 9 to 29% of the trimerous and in 28 to 43% of hemitrimerous seedlings. The average number of intercalary bundles is also generally higher in the dimerous plantlets.

Considering the total bundle number at the base of the hypocotyl (primary bundles plus intercalary bundles) the trimerous and hemitrimerous seedlings have from 0.77 to 1.91 bundles, or from 14.4 to 46.7%, more than the dimerous seedlings. The differentiation of the dimerous and trimerous seedlings is conspicuously shown by the frequency distributions of two of the lines shown in diagram 1.

In passing upward from the base of the hypocotyl, each primary bundle pair normally divides into two so that in the central region of the hypocotyl the bundle number is normally twice the number of primary double bundles at the base, plus the intercalary bundles. In many cases the number is somewhat in excess of this, however, showing either that new (intercalary) bundles have appeared or that some of the bundles have become subdivided.

The modal number of bundles in the mid-region of the hypocotyl is eight or ten in dimerous plantlets; in trimerous and hemitrimerous plantlets it is twelve. On the average the number is from 1.7 to 3.8 bundles higher (or from 15.7 to 47.9% higher) in the trimerous than in the dimerous seedlings. The differentiation of the two classes of seedlings in their vascular anatomy at the level is clearly shown in diagram 2.

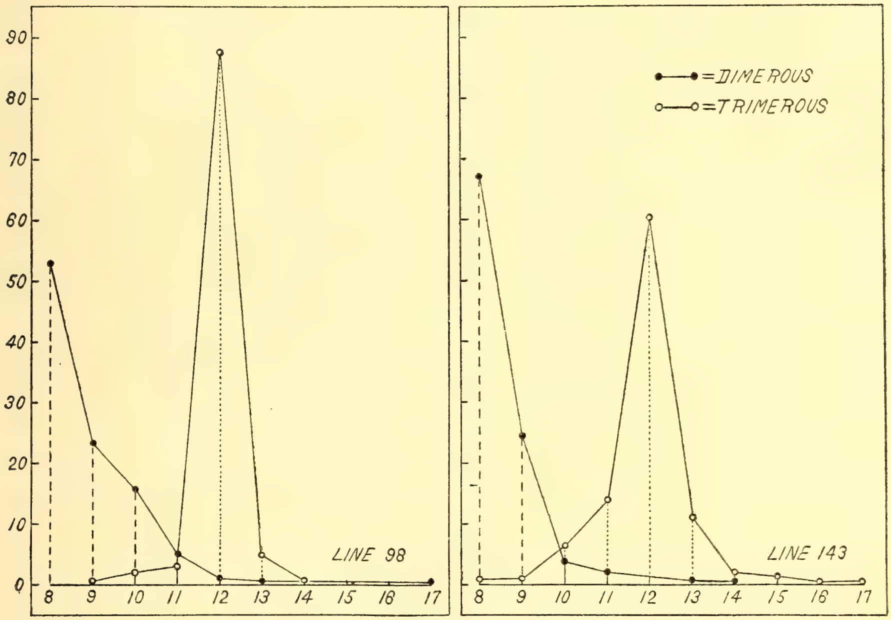

DIAGRAM 1

Percentage frequency distributions of total bundles (primary double bundles counted as two) at the base of the hypocotyl in dimerous and trimerous seedlings of two lines. Abscissae represent bundle numbers, ordinates represent percentage frequencies.

The bundles in the mid-region of the epicotyl show in dimerous plantlets a modal number of twelve, whereas in trimerous seedlings it is fifteen. On the average there are from 2.8 to 3.7, or from 23.0 to 30.2%, more bundles in the epicotyl of the trimerous than in the dimerous seedling.

The form of the frequency distributions for two of the lines is shown in diagram 3. The epicotyl of the hemitrimerous is in essentials of anatomy identical with that of the dimerous seedling.

Not only are there marked differences in the actual number of bundles, but the variability of bundle number changes from region to region of the seedling, and differs in the three seedling types. Whether judged by range, standard deviation or coefficient of variation, the variability of bundle number in the central region of hypocotyl tends to be distinctly higher in the dimerous than in the trimerous plantlets; but in the epicotyl just the reverse is true, the variability of the trimerous plantlets exceeding that of the dimerous. These differences are conspicuous in diagrams 2 and 3. In the first case it is the dimerous plantlets, in the second case it is the trimerous ones which show the greater variability. Apparently this is due to differences in the number of intercalary bundles in the hypocotyl, and to the extent of division of the bundles in the epicotyl, of the two types of seedlings.

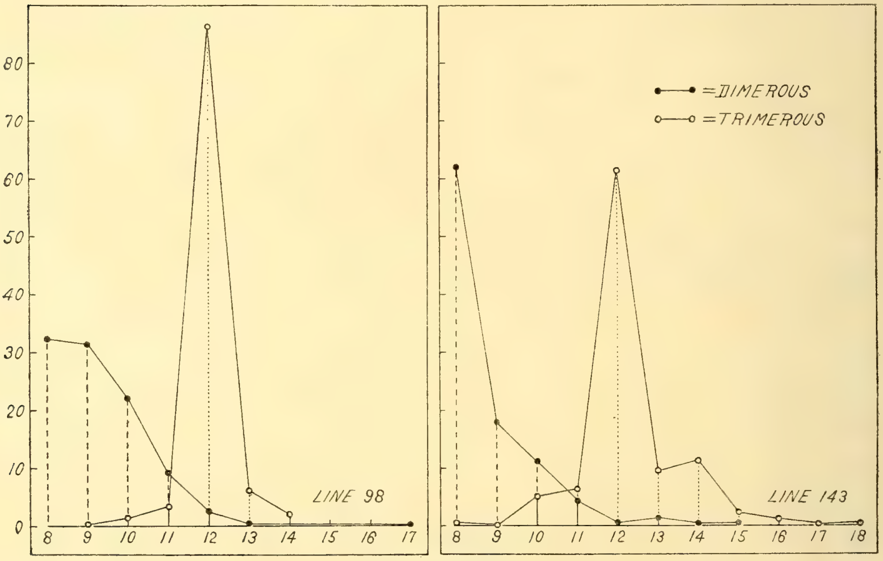

DIAGRAM 2

Percentage frequency distribution of number of bundles in central region of hypocotyl in dimerous and trimerous seedlings. Abscissae represent bundle numbers, ordinates represent percentage frequencies.

The coefficients of correlation between various bundle systems also differ widely. In both trimerous and dimerous seedlings there is a negative correlation between the number of primary double bundles and the number of intercalary bundles at the base of the hypocotyl. Thus the number of intercalary bundles is smaller in seedlings with larger numbers of primary double bundles and vice versa. This result for seedlings of the same (external) morphological type is in agreement with those obtained by a comparison of seedlings which are externally dimerous and trimerous, since the latter frequently have a larger number of primary double bundles but a smaller number of intercalary bundles than the former. In both types of seedlings variation in the number of intercalary bundles is the primary factor in determining variation in the total number of bundles at the base of the hypocotyl.

Turning to the problem of the interrelationship of bundle number at different levels in the seedling we find that there is a substantial correlation between the numbers of the three classes of bundles—primary double bundles, intercalary bundles, and total bundles—at the base of the hypocotyl and the number of bundles in the central region of the hypocotyl. In the normal seedlings the coefficients average +0.509 for number of primary double bundles and number of hypocotyledonary bundles, +0.629 for intercalary bundles and hypocotyledonary bundles, and +0.813 for total bundles and hypocotyledonary bundles. In the trimerous plants these correlations average +0.381, +0.238 and +0.598, respectively. The correlations for normal plantlets are practically without exception higher than those for abnormal seedlings.

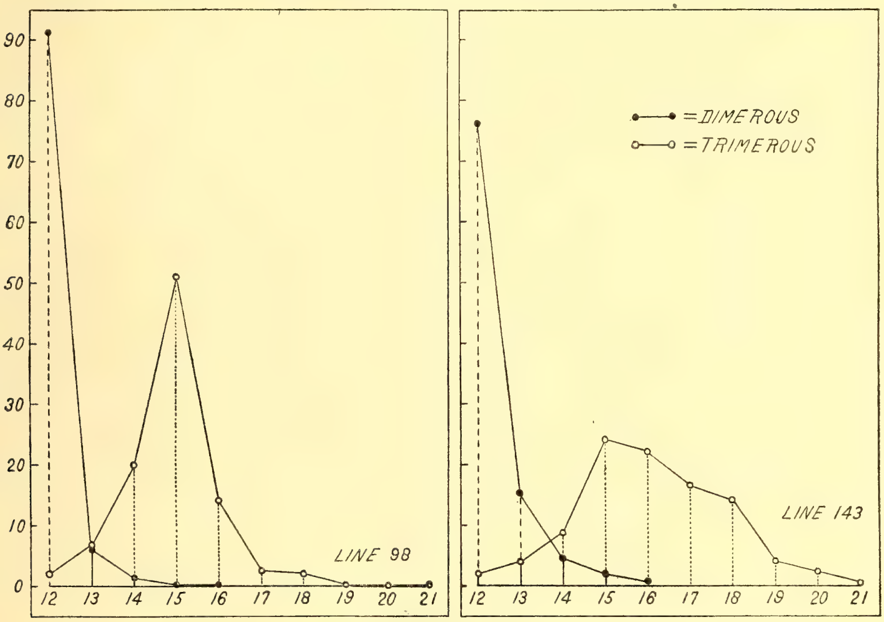

DIAGRAM 3

Percentage frequency distributions of total bundle number in the central region of the epicotyl of dimerous and trimerous seedlings of two lines. Abscissae represent bundle numbers, ordinates represent percentage frequencies.

The correlations between the number of bundles in the hypocotyl (both basal region and central region) on the one hand and the number of bundles in the central region of the epicotyl on the other differ conspicuously from those found between the numbers of bundles at the two levels in the hypocotyl itself, in that the coefficients are extremely slight. They are positive in some series, but negative in others.

The regression straight lines and the empirical means for number of bundles in the hypocotyl (lower line) and epicotyl (upper line) associated with various numbers of primary double bundles at the base of the hypocotyl in two of the lines is shown in diagram 4. The irregularity of the means is at least in part due to the fact that the frequencies are largely concentrated in the two classes of 4 and 5 primary double bundles. The slope of the line representing change in hypocotyledonary bundles associaated with variation in number of basal bundles is steep, showing a rather close dependence; but the line showing the change in epicotyledonary bundles associated with the same variations in basal bundles is very slight, showing the laxness of the interdependence of bundle number in these two regions.

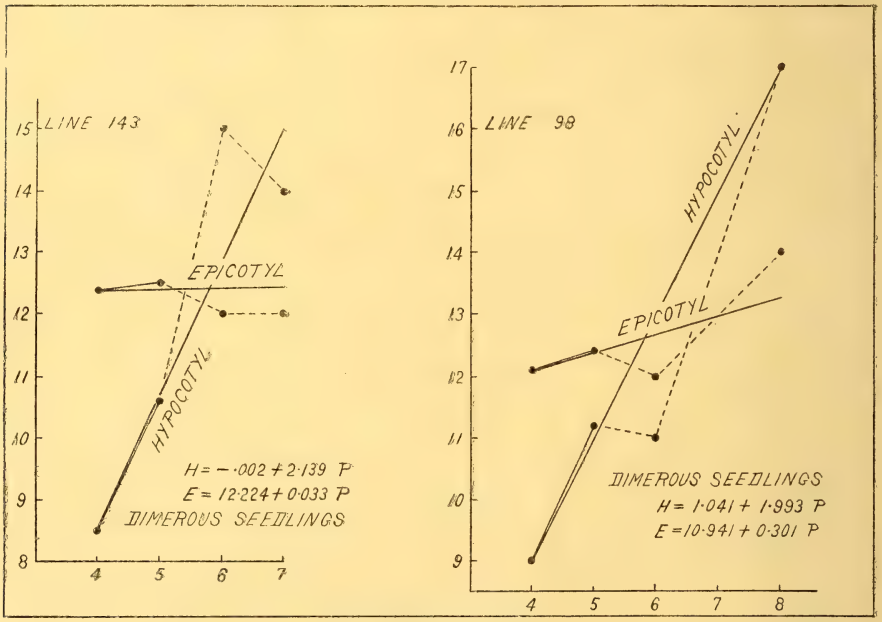

DIAGRAM 4

Regression of number of bundles in central regions of hypocotyl and of epicotyl on number of primary double bundles at base of hypocotyl. Abscissae represent primary double bundles, ordinates represent mean number of bundles in hypocotyl and epicotyl.

On the basis of the data available it is impossible to assert that there is any relationship at all between bundle number above the cotyledons and bundle number below them. These results indicate that there is a *complete reorganization of the vascular system at the cotyledonary node.*

It will be evident from the foregoing outline that the vascular structures of the seedling are not constant but are highly variable, even within genetically very homogenous material. It is quite possible, therefore, that anatomical investigations based on but a few sections for each species might lead to erroneous results. Seedlings differing in external form are differentiated in their internal anatomy. Such differentiation is evident not only in mean number of bundles but in the degree of variability in bundle number. Thus, in normal seedlings the variability is higher in the hypocotyl than in the epicotyl, whereas in seedlings with three cotyledons and three primordial leaves just the reverse is true. The external form and the internal structure of the seedling are highly but not perfectly correlated. Finally in both normal and variant seedlings, the number of vascular elements of the several regions of the seedling are correlated in very different degrees; the correlation between some is high; between others it is practically wanting.

Papers to appear in the *American Journal of Botany* may be consulted for details.

[1] The values given in each case are the maximum and minimum constants for the lines investigated. For the trimerous and dimerous seedlings all the averages are based on five lines. For the hemitrimerous seedlings the averages for primary double bundles and intercalary bundles are based on five lines while for the mid-region of hypocotyl and the mid-region of epicotyl they are based on six lines. Data for number of root poles are available for only three lines.

[2] Because of the extreme rarity of hemitrimerous seedlings it is not feasible to discuss variability of bundle number in this group.

THE PRESENT STATUS OF THE LONG-CONTINUED PEDIGREE CULTURE OF PARAMECIUM AURELIA AT YALE UNIVERSITY

BY LORANDE LOSS WOODRUFF

OSBORN ZOÖLOGICAL LABORATORY, YALE UNIVERSITY

Communicated by R. G. Harrison, December 30, 1920

As a matter of record it seems advisable to bring up to date and summarize the chief results derived from the writer's main pedigree culture of Paramecium aurelia.

This culture, designated Culture I, was started on May 1, 1907, by the isolation of a "wild" individual which was found in a laboratory aquarium.[1] The original specimen was placed in about five drops of culture fluid on a glass slide having a central ground concavity, and when this animal had produced four individuals, each of these was isolated on a separate slide to form the four lines of the culture. The four lines have not been kept distinct from one another throughout the work, but have been replenished by cells from one of the sister lines when, through accident or otherwise, one or another of the lines has become extinct.

The culture has been maintained by the isolation of a specimen from each of these lines practically every day, with the exception noted below. The number of divisions in each line has been recorded at the time of isolation and the average rate of these four lines has again been averaged for varying numbers of days (5, 10, or 30 days) as the exigencies of the different experiments demanded. These data have afforded the graphs of the division rate. Permanent preparations have been preserved from time to time for the study of the cytological changes during the life history. During the first eight months of the work the culture medium consisted of infusions of hay and fresh grass, but from February, 1908, to the present time various materials collected from ponds, swamps, etc., have been employed. The infusions were thoroughly boiled to prevent the contamination of the pure culture with foreign strains of Paramecium. In short, the cells of the four lines of the culture to-day are direct lineal descendants by division of the single animal isolated in 1907.

The object of starting the culture was to determine whether Paramecium can reproduce by division indefinitely without recourse to conjugation. Throughout the work the possibility of conjugation in the four lines of the culture has been precluded by the almost daily isolation of the products of division. Accordingly its continued life and health has long since justified the conclusion that conjugation, involving syncaryon formation, is not, as previously generally maintained, a sine qua non for the continued life of Paramecium, in particular, and, presumably, of Infusoria in general. At the completion of the

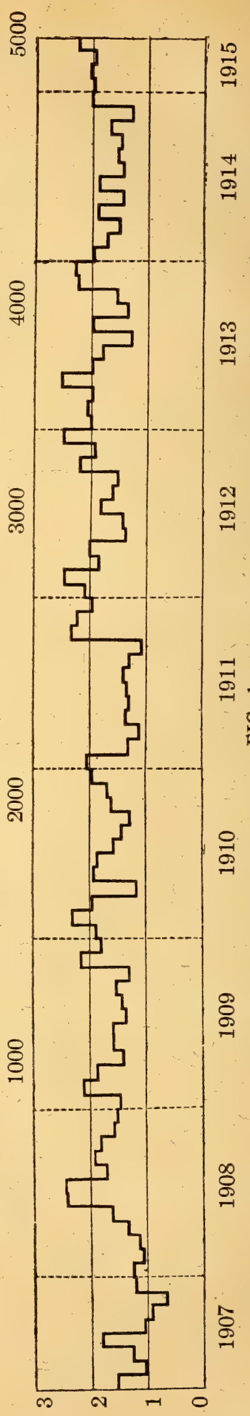

FIG. 1

Graph showing the rate of reproduction of the pedigree race (I) of Paramecium aurelia during the first eight years of culture. The ordinates represent the average daily rate of division of the four lines of the culture, again averaged for each month. The vertical broken lines indicate the limits of the calendar years. The figures 1000, 2000, etc., represent generations and are placed above the periods in which they were attained.

culture's first five years of life, during which over 3029 generations were attained, the writer stated that "the organisms of the present generation are in as normal morphological and physiological condition as the original 'wild' individual isolated to initiate the culture. This study has indicated that, under favorable environmental conditions, the protoplasm of the cell originally isolated possessed (at least) the potentiality to produce similar cells to the number represented by 2 raised to the 3,029th power, or a volume of protoplasm not less than 10^{1000} times the volume of the Earth. I believe this result indicates beyond question that the protoplasm of a single cell may be self-sufficient to reproduce itself indefinitely, under favorable environmental conditions, without recourse to conjugation......"

As a matter of fact organisms taken from this culture of Paramecium have shown but little tendency to conjugate. From time to time experiments to induce conjugation have been carried out in mass cultures started from the pedigree lines. The first successful experiment occurred in December, 1913, at about the 4100th generation.[2] The next epidemic of conjugation in mass culture occurred June, 1920.

Careful studies of the rate of division of sub-lines derived from the main lines and bred under the most constant environmental conditions revealed the fact that there are inherent, normal, minor, periodic rises and falls of the fission rate due to some unknown factor in cell phenomena. These were termed Rhythms.[3] In a search for the underlying factors of rhythms a complicated internal nuclear reorganization process was discovered and named Endomixis.[4] This process involves the periodic formation of a complete new nuclear apparatus by a definite sequence of normal morphological changes which simulate those occurring in conjugation. Endomixis, in essence, consists of a gradual disintegration and absorption of the macronucleus in the cytoplasm. Simultaneously a multiplication of the micronuclei is in progress. Certain of the resulting micronuclei degenerate while the remaining one or two form the new macronuclear and micronuclear apparatus of the cell. This results in the

YEAR	GENERATIONS	DAILY AVERAGE DIVISION RATE
1	452	1.24
2	690	1.89
3	613	1.68
4	612	1.67
5	662	1.81
6	692	1.89
7	671	1.84
8	679	1.86
Average for 8 yrs.	634	1.74

reorganization of the cell (endomixis) without the coöperation of two animals involving syncaryon formation (amphimixis) as in conjugation.

After endomixis was worked out, it seemed unnecessary to continue longer the main pedigreed culture and accordingly on May 1, 1915, at the 5071st generation, the experiment was *formally* considered as closed.

The record of divisions for each year of the life of the culture to May 1, 1915, may be tabulated as shown in the preceding table.

Thus for eight years the culture averaged about fifty generations per month. However, at the formal termination of the experiments in 1915, I was reluctant to discard the race and was tempted to keep it under control but without such exacting daily observation and recording of the daily division rate. In this manner it has been maintained up to the present (December, 1920). But, from time to time, thirty-day tests have been made of the vitality of the race under the former rigid culture conditions, and in each case the same general average division rate has been revealed as during the first eight years of its life. For example, during December, 1915, 52 generations; during December, 1917, 56 generations; and during November, 1920, 54 generations were attained. On the basis of such tests it is fair to make the exceedingly conservative estimate of 600 generations attained each year since May 1, 1915. This would give, in round numbers, 8400 generations attained by the culture during the $13^{1}/_{2}$ years of its life to date.

Thus the conclusion is still justified that conjugation is not a necessary phenomenon in the life history of Paramecium aurelia under favorable environmental conditions. But there is an internal reorganization process (endomixis) which occurs periodically. Whether endomixis is a necessary factor for the continuance of the race is another question—a new question which has been raised by these studies, and is now under investigation.[5]

[1] Woodruff, L. L., "The life cycle of Paramecium when subjected to a varied environment," *Amer. Naturalist*, **42,** 1908. "Two thousand generation of Paramecium," *Archiv f. Protistenkunde*, **21,** 1911.

[2] Woodruff, L. L., "So-called conjugating a non-conjugating races of Paramecium," *J. Exper. Zool.*, **16,** 1914.

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[4] Woodruff, L. L., and Erdmann, R., "A normal periodic reorganization process without cell fusion in Paramecium," *J. Exper. Zool.*, **17,** 1914.

[5] Woodruff, L. L., "The problem of rejuvenescence in protozoa," *Biochem. Bull.*, **4,** 1915; "The influence of general environmental conditions on the periodicity of endomixis in Paramecium aurelia," *Biol. Bull.*, **33,** 1917.

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POSITIVE RAY ANALYSIS OF MAGNESIUM

By Arthur J. Dempster

Ryerson Physical Laboratory, University of Chicago

Communicated by A. A. Michelson, Dec. 7, 1920

In the *Physical Review* for April, 1918, I described an apparatus for positive ray analysis and gave examples of hydrogen, oxygen, sodium and potassium rays, showing that at least the lighter elements could be readily analyzed so as to separate any molecules differing in molecular weight by unity. I have recently resumed the experiments and will give in this paper an account of experiments with magnesium.

As stated in the *Physical Review* article, the experimental difficulties are largely in obtaining a steady source of the rays desired. Magnesium rays have been obtained from a piece of the metal which was heated electrically by a coil of wire, and at the same time bombarded by electrons from a Wehnelt cathode. The occluded gases are first driven off, and then the heating current is increased till the magnesium lines appear due to the metal vaporizing slightly. The positively charged molecules formed pass through a hole in a plate below the cathode and are then accelerated by a strong variable field of several hundred volts. The plate has been added to the apparatus described in the *Physical Review* to prevent the strong electrical field influencing or even inhibiting the low voltage discharge. The first slit, about 1 mm. wide, separates out a bundle of rays which is bent into a semicircle by a strong magnetic field and refocussed, if their speed is right, on the second slit, below the detecting electrode. The charge carried by the rays, which is proportional to the number of the molecules of different kinds, is measured by a Wilson electroscope used as a null instrument with a special compensating device for rapid measurements.

The charged atoms of different atomic weights are successively brought on to the detecting electrode by keeping the magnetic field constant and varying the potential which accelerates the rays, the potential required being inversely proportional to the mass of the particles. Thus, if one atomic weight is known the others may be found. Due to the finite width of the slits, each element gives a curve, on the atomic weight scale, which is theoretically a linear increase to a maximum and then a linear decrease. The width half way to the maximum is given by $m. \frac{2S}{d}$ where m is the atomic weight, S the slit width and d the diameter of the circle in which the rays travel. Under good vacuum conditions this theoretical sharpness is practically obtained. For 1 mm. slits this width of the curves should thus be one-half a unit on the atomic weight scale. The former measurement with the apparatus and the magnetic field determinations sufficed to locate elements between 20 and 30 within one unit, and identified the strong nitrogen rays (possible carbon monoxide) of molecular weight 28 which are given off when the metal is first heated.

One series of experiments was as follows: After heating the magnesium slightly and pumping, till a MacLeod gauge gave no pressure indication, the nitrogen molecule was the only particle present. The heating current was then increased by steps to vaporize the magnesium. With 0.7 ampere, 28 alone was present, with 0.75 ampere an arc apparently struck as the cathode-anode current jumped suddenly to five times its value. The electron current was decreased to its former value by cooling the cathode and the rays were measured. It was found that three strong new lines had appeared. The new lines which are undoubtedly due to magnesium were compared with the nitrogen rays which were still faintly present and found to have atomic weights 24, 25 and 26. The observations are illustrated in figure 1, which gives the current or number of particles for different atomic weights. The nitrogen line had its maximum at 817 volts, and the atomic weight abscissae are 28 × 817 divided by the volts applied. The ordinates of the 28 line are multiplied by 10 in plotting to make them comparable with the other three lines. The dotted continuation to the axis indicates the slight overlapping of the lines. We conclude that magnesium consists of three isotopes of atomic weights 24, 25 and 26.

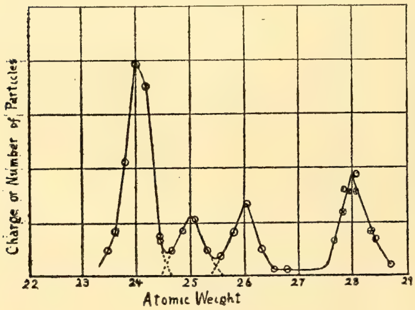


FIG. 1

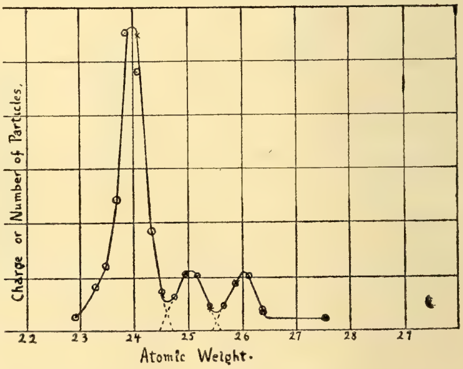


FIG. 2

Later curves made with steadier discharge conditions are more suitable than figure 1 for measuring the relative strengths of the components. In figure 1 there appears to have been a drop in intensity just before 24 was reached, in the measurement from high to low atomic weights. The curve is of interest as still containing 28 faintly and so serving to accurately locate the weights which otherwise would have been uncertain to a fraction of a unit.

Figure 2 is one of several later curves taken under steadier conditions. These all have very closely the same appearance. The components 25 and 26 are present very nearly in equal amounts; in some measurements 25 was found about nine-tenths the intensity of 26. The component at 24 is approximately 6 times as strong as the one at 26. The ratio of 1 : 1 : 6 gives an average atomic weight 24.375, which is in as good agreement with the accepted atomic weight for magnesium as could be expected with the wide slits used in these first experiments.

THE ENERGY CONTENT OF THE DIAPASON

By CARL BARUS

DEPARTMENT OF PHYSICS, BROWN UNIVERSITY

Communicated January 10, 1921

In *Science*, **52,** 1920 (586–8), I indicated a method by which the intensely luminous achromatic fringes could be used, without further mechanism, to determine the compression in the interior of a sounding organ pipe.

Meanwhile Profs. A. T. Jones, of Smith College, and H. F. Stimson, of the Bureau of Standards, have called my attention to papers of Boltzmann (*Pogg. Ann.*, **141,** p. 321) and Raps (*Wied. Ann.*, **50,** p. 193) and to some work of Stimson himself, which I had overlooked. These researches make most of my work superfluous. I will, therefore, confine myself to a few special features, as the interferometer which I set up, admitting of any separation of the interfering beams, longitudinally or laterally, is better adapted for work of this character than the Fresnellian fringes or the Jamin interferometer used heretofore. Two opposed nodes may be examined simultaneously. Moreover the ease with which fringes of any size or inclination are producible is a further advantage.

As the transformations are adiabatic, the density increment $\Delta\rho$ at any time is of the form $\Delta\rho = C.n\lambda/lR$, if n fringes of wave-length λ are displaced when the ray passes through a pipe of length l. R is the gas constant and

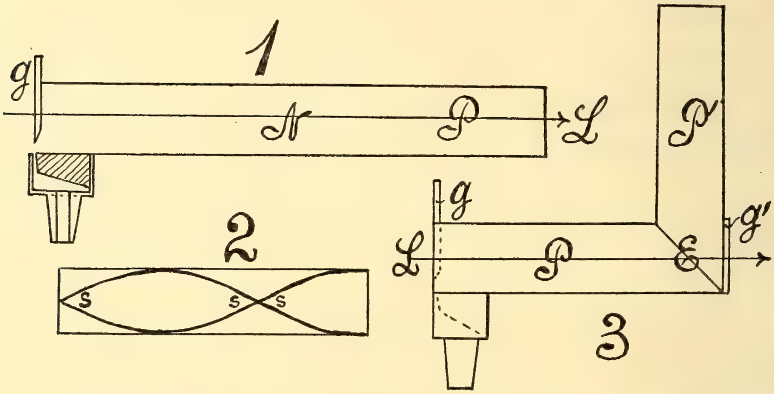
$$C = \frac{p_0}{\vartheta_0(\mu_0 - 1)} = 10^7 \times 1.27$$

the optic constant when p_0 and ϑ_0 are standard pressure and absolute temperature, and μ_0 the corresponding index of refraction of air. Thus the mean mechanical energy per cm.^{3} is $p\Delta\rho/\rho$ for the length l surveyed, and if this comprises $\lambda/2$, the maximum energy will be $\pi/2$ times larger. Thus the maximum work spent in compressing a cm.^{3}, for n fringes between trough and crest, if $p = 10^6$, $\lambda = 60 \times 10^{-6}$, $\rho = 0.00129$, $l = 24$ cm., $R = 10^6 \times 2.87$, is

$$p\,\Delta\rho/\rho = \frac{\pi p C \lambda}{2\rho l R}\,n = 10^3 \times 13.7\,n$$

and the compression, $\Delta\rho/\rho = 10^{-3} \times 13.7\,n$.

Observations with longitudinal pipes.—The open pipe, figure 1, was of wood, with an inner cross-section 2.8 × 2.8 cm.^{2} and length 24 cm. It responded perfectly to the fundamental, the softest producible note showing about 0.1 fringe in double amplitude, the loudest full note not more than a whole fringe. Blown so hard that the note sharpened, the amplitude rather diminished. As in the case heretofore described, the mean energy lies between ($l = 24$ cm. here)



$$p d\rho/\rho = 10^3 \times 0.87 \text{ ergs/cm.}^3$$

and ten times this quantity. The maximum energy would, therefore, be $\pi/2$ times larger, which makes the energy content $10^4 \times 1.37$ ergs/cm.^{3} for the full note.

The pipe did not at first admit of the production of the octave free from the fundamental. The wave pattern consisted of fundamental waves along the contours of which octave waves passed to and fro. The embouchure was, then, reset to give the clear shrill overtone alone. The waves now vanished evidencing complete symmetry in the dense and rare nodes present; proving, moreover, that the interferometer contributed nothing.

Similar results were obtained with the closed pipe. The surprising feature of this closed pipe, however, was the occurrence of strong waves for the first overtone (fifth above octave). The two nodes, therefore, are here not symmetrically rare and compressed. The double amplitude of these waves for the very shrill note was 0.3 to 0.4 fringe. Neither the waves nor the sound gave any indication of the fundamental. Thus the node at the closed end of the pipe is a semi-node S, figure 2, while that nearer the open end is a full node SS, and the difference found is equivalent to the mean energy of the former and should be trebled. The duodecimal overtone was also obtained sharply.

Finally the temperature equivalent of the mechanical energy per fringe, computed either as $Jc\rho\Delta t = p\Delta\rho/\rho$, or as $\Delta t = \tau(k/c-1)\Delta\rho/\rho$ comes out 1.5° C., if $\tau = 273°$. Some years ago I tried to measure this with a bolometer-telephone device, but failed. This reason seems to be that in case of these rapid alternations, heat does not enter the wires appreciably.

Reed Pipes. Voice.—In the endeavor to obtain waves of larger amplitude, the device resorted to was a brass pipe 32 cm. long and 3 cm. in diameter, through which the component ray of the interferometer passed, provided with a tubulure about 1 cm. in diameter at its middle. This was then connected with flexible tubing to the reed box. The note was coarse like a bassoon.

With the flexible connector very short, the first experiments gave a display of enormous waves, 20 or 30 fringes high. I suspected that this could only be a direct mechanical effect of the sonorous reed on the interferometer. The reed pipe was, therefore, mounted on a separate scaffolding, entirely independent of the interferometer. The result was an immediate reduction of double amplitudes to a few fringes. On replacing the reed by an clarionette mouth piece, the results were similar.

The pipe was now moved, as a whole, out of the range of the rays of the interferometer and sounded. To my astonishment, strong waves were again produced, nearly as much so as when the pipe was in position for interference. In other words, these reed notes act directly on certain parts of the interferometer, and excite the parts selected by resonance.

To test this further, I made use of the voice, singing a foot or more away from the interferometer. At certain chest notes (b, c'), the fringe bands broke into marked waves near a fringe in double amplitude, the effect being absent from the remainder of the scale. A clarionette played about a yard or more from the interferometer evoked the following response:

$e..g$	a	b	c'	d'	e'	f'	g'	$a'..f''$
0..0	strong	stronger	Max. 1.2 fringes	strong	weak	0	weak	no response

Resonating pipes on the interferometer had no discernible effect. The seat of receptivity is probably the iron base of the apparatus. Loading it depressed the maximum to b. A totally different interferometer, in a new location, showed the same behavior on the same base (lathe bed). In a third interferometer of different construction and on a different base, the clarionette e was most effective, b, c', f', a' marked the remaining notes ineffective. I also constructed diapason pipes over 60 cm. long and 5 cm. in diameter, which were excited with my adjustable embouchure. The full c' obtained had the same direct effect on the interferometer as the clarionette. This discrepancy is exceedingly difficult to eliminate as it calls for a detection of the resonant member of the interferometer.

With the 1-foot diapason organ pipes used above, there is much less danger of direct influence. This is shown, for instance, in the balance obtained with nodes of opposite sign. Moreover, I made control experiments by blowing equipitched diapason pipes strongly in the neighborhood. There is even here liable to be a little response. The tendency to assume wave form may be recognized; but it is much smaller than the pipe note proper, and quite absent in the overtones. Finally, the elbowed pipe, figure 3, which blows away from the interferometer, was used for additional guarantee and for overtone nodes.

A PRELIMINARY NOTE ON THE RESULTS OF CROSSING CERTAIN VARIETIES OF NICOTIANA TABACUM

By William Albert Setchell, Thomas Harper Goodspeed and Roy Elwood Clausen

Department of Botany and Division of Genetics of the Department of Agriculture, University of California

Communicated January 10, 1921

In connection with a taxonomic study of the various species and varieties of *Nicotiana*, the authors became interested in the extremely varied assemblage of varieties, both botanical and commercial, included under the species *N. Tabacum*. The senior author ventured to suggest a preference for five type varieties as representative of the range of variation found within the species and possibly of fundamental importance as stem forms in the derivation of other varieties. A similar attempt to refer existing commercial varieties to derivation from a limited number of fundamental forms had previously been made both by Comes[1] and Anastasia.[2] These authors agreed in principle on the method of derivation of existing varieties, but they held conflicting views as to which particular forms should be recognized as fundamental. In all three cases the principle followed in attempting to unravel the problem of origin of cultivated forms was to determine which few historically old varieties possessed in various combinations all the characters exhibited by commercial varieties, and then to refer existing varieties to hybridization with resulting segregation and recombination of characters exhibited in the stem forms.

The senior author, having tentatively selected five such stem forms, thought it wise by actual genetic experimentation to determine what results would follow hybridization among them. The authors also found themselves in need of some definite information as to the Mendelian details involved in character differences within the species. Accordingly three crosses were actually made and studied at some length. In some cases derivative lines were carried into the tenth hybrid generation. The three crosses, all in reciprocal, were made in 1909 as follows: (1) *angustifolia x macrophylla*, (2) *calycina x virginica* (Maryland) and (3) *alba* (White) *x macrophylla*. Detailed descriptions of the above-named varieties have been given by Setchell[3] in a previous paper. In this article we shall confine our attention to the Mendelian results of the studies.

The Angustifolia-Macrophylla Series.—In the *angustifolia-macrophylla* series an attempt was made to subject two character differences to Mendelian analysis, viz., the light pink flower color of *angustifolia* versus the red of *macrophylla* and the long-petioled leaf-base of *angustifolia* versus the sessile, broad leaf-base of *macrophylla*. The former of these contrasts proved to be a simple monogenic one, but the latter necessitated a complex Mendelian formulation.

Turning our attention to flower color first, F_1 was pink-flowering but the intensity of coloration was distinctly greater than that of *angustifolia*. Ten F_2 families containing a total of 490 plants consisted of 377 pink-flowering and 113 red-flowering plants. The pink-flowering segregants clearly gave evidence of the existence of two sub-classes differing in intensity of coloration. The count totalled 240 pink-flowering and 137 light pink-flowering individuals, a rough approximation to a 2 : 1 ratio. Very little reliance, however, can be placed upon this classification because of the evident intergrading of the two sub-classes. Separation into red and pink was clear and distinct. The F_3 populations conformed to expectation based upon the F_2 results, with the single exception noted below. Four F_2 red-flowering selections gave 99 F_3 plants all red-flowering. Of fifteen F_2 pink-flowering selections ten gave progenies exhibiting segregation in conformance with the 3 pink: 1 red ratio, the actual totals being 187 pink: 60 red. Four F_2 pink-flowering selections bred true for pink, the entire 89 F_3 plants being pink. One population from an F_2 pink-flowering selection yielded a progeny consisting entirely of red-flowering plants. It is difficult to state whether this result indicates a misclassification of the F_2 parent or whether some other experimental error was to blame for it. Taken as a whole the data adequately establish the monogenic nature of the pink : red character contrast. Light pink is to be considered as incompletely dominant over red, and the genetic difference may be designated by the symbols R for light pink and r for red.

The analysis of the leaf-base difference proved to be rather complicated. F_1 was petioled, but the petiole was shorter than that of *angustifolia* and the wing of the petiole was more amply developed. In F_2 a hopelessly complex series of forms was obtained ranging from extremely long-petioled to very broadly sessile types. Although it was impossible to demonstrate any valid Mendelian distribution in F_2, it was possible to distinguish certain forms as centers of distribution. Twenty F_2 selections with respect to the character of the leaf base were made and their progenies were studied in F_3 and subsequent generations. The twenty selections fell into six general leaf types, the names and descriptions of which follow:

1. *Stenophylla,* long-petioled like *angustifolia,* and with a minimum wing development along the petiole.
2. *Latifolia,* short but distinctly petioled like the F_1, and with a variable wing development.
3. *Lanceolata,* broad-based sessile leaves with long lanceolate blades.
4. *Loriifolia,* sessile leaves with very narrow strap-like blades.
5. *Auriculata,* leaves with the blades sharply constricted at the base, nearly if not quite to the mid-rib, and typically with clasping auricles. In some *auriculata* forms the auricles may be absent, in which case the leaf may appear to have a short distinct petiole.
6. *Sessilifolia,* sessile leaves with a broad base and a broad clasping insertion.

Of the above six types of leaves, three, *lanceolata, loriifolia,* and *sessilifolia,* differ merely in width of blade. Genetically they appear to be identical in their leaf-base factors. For the purposes of this discussion, therefore, it is only necessary to distinguish four primarily leaf-base types, *stenophylla, latifolia, sessilifolia,* and *auriculata.*

As we have stated, F_2 gave no decisive evidence either as to the number of factors involved in the leaf-base contrast or as to the interactions among them. In order to obtain suitable analytical material we were obliged to study populations of F_3 and subsequent generations. In such populations a notable simplification was observed in the number of segregation products, and the distribution into classes was more distinct. In the course of the investigations, also, constant derivatives of the various leaf-base types mentioned above were established, and further information as to the genetic relation of the different types was obtained by intercrossing these lines and by crossing them with the original parents. Theoretically such lines should differ less from the parents and from one another than the original parents did from each other, and practically this was found to be the case. The behavior in skeleton of selections of various leaf-base types is given below.

Stenophylla selections were observed to segregate in a variety of ways. Two populations bred true for *stenophylla;* three segregated into 3 *stenophylla:* 1 *sessilifolia;* and one gave 3 *stenophylla:* 1 *auriculata.* A *latifolia* derivative crossed with *angustifolia* gave F_1 *stenophylla,* and F_2 3 *stenophylla:* 1 *latifolia.*

Latifolia selections proved rather perplexing. In some cases *latifolia* is evidently a complex hybrid expression, as is evidenced by the fact that the F_1 cross between the original parents was typically *latifolia* in appearance. One selection repeated the complex F_2 segregation in F_3; two bred true for the *latifolia* leaf-base type; one gave a fairly definite segregation into 3 *latifolia:* 1 *sessilifolia* with subsequent establishment of both types in constant derivative lines; and one population gave 3 *latifolia:* 1 *sessilifolia.*

Sessilifolia selections behaved very simply. Of nine such selections, five bred true for *sessilifolia,* and the remaining four gave 3 *sessilifolia:* 1 *auriculata.*

Auriculata selections appear to breed true whenever tested. One population is an exception to this statement, but it was unfortunately not further investigated. Two other selections bred true for *auriculata.* An F_5 light pink *auriculata* derivative crossed with *macrophylla* (red *sessilifolia*) gave pink *sessilifolia* in F_1, and 133 pink *sessilifolia:* 40 pink *auriculata:* 53 red *sessilifolia:* 16 red *auriculata* in F_2, a very close approximation to an independent dihybrid expectation.

On the basis of these facts we have been bold enough to offer provisionally the following formulation for the major differences in leaf-base characters:

Ss, stenophylla versus *sessilifolia, SS* is long-petioled like *angustifolia, ss* broadly sessile like *macrophylla.* The heterozygote may possibly approach an intermediate condition similar to *latifolia.*

Ll, stenophylla versus *latifolia. SSLL* is long-petioled like *angustifolia;* and *SSll,* short-petioled like *latifolia,* and with a distinct but not broad wing on the petiole. *Latifolia* is evidently a modified petiolate form.

Aa, sessilifolia versus *auriculata, ssAA* has the broad clasping leaf-base characteristic of *macrophylla;* and *ssaa* the deeply constricted leaf-base with flaring auricles of *auriculata. Auriculata* is clearly a modified sessile form.

As to the completeness of dominance in any of these factor pairs, we can say very little because of the difficulty of evaluating the effect of the residual genotype. Likewise the significance of the wing in *latifolia* or of the auricles in *auriculata* is not definitely established on account of the extreme genetic difference which existed between the original parent varieties.

The Calycina-Virginica Series.—In the *calycina-virginica* (Maryland) series most of the attention was given to flower color and flower form. The flower color of *calycina* was red like that of *macrophylla,* and that of *virginica* was light pink like that of *angustifolia.* The results were in full agreement with those obtained from that series. The F_1 was pink and F_2 gave 73 pink: 23 red, with the same variation in the pink class as that noted for the preceding series. Five red F_2 segregants gave 125 F_3 plants all red-flowering. Of twelve F_3 families grown from pink segregants, three gave only pink-flowering descendants, and nine exhibited segregation into 3 pink: 1 red, the actual totals being 170 pink: 52 red.

Calycina is characterized by the production of split hose-in-hose flowers. The calyx is strongly developed and petaloid and it is also deciduous in contrast to the persistent and at times accrescent nature of the calyces of normal forms. This character proved to be a simple recessive. F_1 was normal except for slight suggestions of a petaloid condition of the calyx tips on some of the flowers of the F_1 plants. F_2 gave 61 normal: 35 split hose-in-hose. In F_3, seven families from F_2 split hose-in-hose segregants gave 175 plants all bearing split hose-in-hose flowers. Of ten families from normal F_2 segregants two gave only normal descendants, while eight exhibited segregation into normal and hose-in-hose in the numbers 133 normal: 42 split hose-in-hose. Four F_2 families were examined for simultaneous segregation of pink versus red and normal versus split hose-in-hose. The results were 109 pink normal: 36 pink hose-in-hose: 43 red normal: 11 red hose-in-hose, which is in substantial agreement with a 9 : 3 : 3 : 1 ratio for independent segregation in these two pairs of allelomorphs. The results establish the existence of a single pair of allelomorphs, *Cc*, for the character contrast normal versus split hose-in-hose (calycine) flowers.

The Alba-Macrophylla Series.—In the *alba* (White)-*macrophylla* series our attention was largely confined to flower color, *alba* being white and *macrophylla,* as we have seen, red. The F_1 in this case was definitely pink, and F_2 gave (combining 1911 and 1916 results) 180 pink: 58 red: 83 white. This result is in very satisfactory agreement with a 9 pink: 3 red: 4 white ratio, which may be accounted for by adding to the *Rr* pair of allelomorphs previously demonstrated a second pair, *Ww,* for colored versus white flowers. *Alba* is then *RRww,* and *macrophylla, rrWW;* and the F_1, *RrWw,* is pink as in the two previous series. Of five F_3 families grown from red F_2 segregants, three (*rrWW*) bred true for red; one (*rrWw*) gave 35 red: 14 white; and one gave the anomalous segregation product of 2 red: 23 white. Seven families were grown from pink F_2 plants. Five (*RrWw*) segregated for pink, red, and white in the numbers 73 pink: 25 red: 28 white; one (*RrWW*) gave 22 red: 3 pink; and one (*RRWw*) gave 33 pink: 9 white. Only one pink genotype, *RRWW,* was not demonstrated, but this is not inexplicable in view of the fact that only one in nine of the F_2 pink-flowering plants should belong to this genotype. Seven F_3 families were grown from white F_2 segregants. Among 191 plants, otherwise white-flowering, there was a single pink-flowering individual. It was doubtless a stray of some sort. The difference between the red flowers of *macrophylla* and the white ones of *alba,* therefore, finds a satisfactory formulation as a bigenic contrast.

Discussion of Results.—The results of the flower color studies set forth above appear at first sight to conflict with those which have been published by Allard.[4] He found that carmine, a color only slightly different from the red of *macrophylla,* was a simple dominant to pink; and when he crossed it with white, he obtained F_1 lighter than carmine, and an F_2 progeny which might have been interpreted as 9 carmine: 3 pink: 4 white. Certain back crosses which he made and out crosses of extracted whites to pink bear out this analysis. Of his varieties his pink (Maryland Mammoth) appears to be identical with our *virginica* (Maryland), consequently we are forced to assume that his carmine is genetically distinct from our red. It follows that if we add to the two pairs of factors for flower color which we have demonstrated, a third, *Pp*, for carmine dominant versus pink recessive, his results are brought into accord with ours. Certain experimental results which we have just secured and which we are reserving for future treatment demonstrate the correctness of this suggestion.

A general result of these investigations has been a demonstration of the complexity of difference from a genetic standpoint between any two of these so-called fundamental varieties of *Tabacum*. In one sense this result confirms the opinion of Comes and of Anastasia as to the manner of origin of the vast assemblage of *Tabacum* varieties. It goes further, however, and demonstrates the futility of seeking to determine affinities on the basis of morphological studies unaccompanied by experimental investigations. Thus, for example, we have seen such puzzling segregation products as the *auriculata* and *loriifolia* leaf types, the former with its peculiarly constricted leaf base, the latter with a much narrower leaf blade than either parent, coming out of the *angustifolia-macrophylla* series. Moreover, the demonstration of the existence of genetically distinct red-flowering varieties is another evidence of the limitations of purely morphological studies and of the errors in the determination of affinities to which such studies are subject. The genetic studies outlined above indicate merely that *Tabacum* is a group species like corn, barley, oats, etc., possessing a complex series of allelomorphic contrasts. The so-called fundamental varieties of *Tabacum* intercross freely and produce fully fertile progenies. They cannot genetically, therefore, be regarded as representing anything but a few very distinct genotypes. A demonstration that a few such varieties may contain within them the possibility by means of recombination of producing a host of secondary varieties does not really demonstrate that they are fundamental. In fact when we consider the fugitive nature of our *Tabacum* varieties, except in so far as they are kept isolated by natural or artificial means, the conclusion appears inevitable that we must regard all our varieties as fundamentally equivalent from a genetic standpoint. The really significant problem in considering the species is the determination of how these allelomorphic contrasts have come into existence. These investigations throw no light upon that problem.

We have been interested in the methodology of Mendelian analysis in *Tabacum*. Doubtless one of the reasons why our knowledge of the details of inheritance in *Tabacum* is so meager is because of the prevailingly quantitative or semi-quantitative nature of the character differences. Even in the case of flower color, the impression given by a series of varieties arranged in order of intensity from white to intense dark red is one of graduated differences in amount of a single pigment rather than in qualitatively different pigments. A definite alternative analysis can, however, be made even for these semi-quantitative characters by growing successive generations of segregating progenies until the progenies have been freed of segregation products other than those which it is desired to analyze. Furthermore, the establishment of constant derivative races and the subsequent study of intercrosses among them has been found to result in simplification of the difficulties of analysis. Both of these methods evidently depend upon stabilizing the residual genotype, which is a prime desideratum in the accurate analysis of semi-quantitative characters. The fact that such simplification of segregation can be accomplished and that semi-quantitative characters may then be subjected to analysis according to the qualitative mode of procedure argues not only for the adequacy of Mendelian principles in these cases, but for the identity in principle of qualitative and quantitative characters.

The experimental data cited above were obtained from cultures made possible by a portion of the Adams' Fund allotted to the Department of Botany by the Department of Agriculture of the University of California. The detailed account of this series of studies will appear in a forthcoming number of Vol. 5 of the University of California Publications in Botany under the title, Studies of inheritance in Nicotiana Tabacum, I. A report on the results of crossing certain varieties.

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"Della Razze dei Tabacchi Filogenesi, Qualita ed Uso," *Ibid.*, **57**, 1905.

[2] Anastasia, G. E.: Le varieta Tipiche della *Nicotiana Tabacum* L. Scafati, 1906.

[3] Setchell, W. A.: "Studies in *Nicotiana*, I," *Univ. Calif. Publ. Botany*, **5**, 1912 (1–88).

[4] Allard, H. A., "Some studies in blossom color inheritance in tobacco, with special reference to *N. sylvestris* and *N. Tabacum*," *Amer. Naturalist*, **53**, 1919 (79–84).

THE CORRELATION OF COMPOUND FORMATION, IONIZATION AND SOLUBILITY IN SOLUTIONS. OUTLINE OF A MODIFIED IONIZATION THEORY

By JAMES KENDALL

CHEMISTRY DEPARTMENT, COLUMBIA UNIVERSITY*

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Introduction.—The anomaly of strong electrolytes has been a notoriously weak point in the ionization hypothesis ever since its inception. In fact, satisfactory agreement with the Ostwald dilution law (which follows from the application of the law of mass action to the ionization equilibrium $RX \rightleftharpoons R^+ + X^-$) has been definitely established only for two restricted types of electrolytes (weak acids and bases) in a single solvent (water) through a limited concentration range (dilutions greater than $N/16$).

Numerous attempts towards reconciliation having failed, the tendency is now wide-spread either to deny the applicability of the law of mass action to the ionization of strong electrolytes or to modify the ionization equation itself.

Complete ionization of strong electrolytes, with an electrostatic equilibrium between "free" and "bound" ions, has recently been urged by Milner, Ghosh and others. The existence of inter-ionic attractive forces and the necessity for recognizing their effect in any accurate ionization formula cannot be questioned, yet there are several points of difficulty which the Milner-Ghosh conception still leaves unexplained. Among these may be mentioned the lack of a definite gap between strong and weak electrolytes and the fact that the heat of ionization (as ordinarily defined) does not agree at all with the work required for the separation of the charged radicals. It is quite possible, indeed, that the concordance obtained by Ghosh between experimental and calculated "ionization" values may ultimately prove to be due rather to convenience of constants than to the fundamental validity of the equations developed. Full discussion of these points must be reserved for a future article. A still more serious objection, however (which applies also to the original hypothesis of Arrhenius), is the neglect of the rôle of the solvent in ionization.

The essential equivalence of solvent and solute in ionization equilibria has been elaborated upon in a previous paper.[1] We take two liquids; for example, water and (liquefied) hydrogen chloride. Both are highly associated, both have high dielectric constants, both possess practically zero conductivity when perfectly pure. We mix them and obtain a solution with very high conductivity, which we ascribe to the dissociation of the hydrogen chloride into its constituent ions. Is there really any reason why the conductance should be referred entirely to hydrogen chloride and not at all to water? The *active* part played by the solvent in ionization is unconsciously recognized by our admission that ions are "hydrated" in aqueous solution, since (for example) a "monohydrated chlorine ion" $(Cl,H_2O)^-$ might just as conceivably deserve the alternative title of a "solvated hydroxyl ion" $(HCl,OH)^-$. A scheme according to which water is the only "ionizing" substance in all aqueous solutions would, in point of fact, be quite as logical (and certainly as simple) as the scheme at present employed, which relegates water to the inferior rôles of "dead space" and "envelopes." Much more satisfactory than either, however, would be a theory taking both components of a conducting solution equally into account.

Considerable experimental work, upon which such a theory may be based, has been performed by the author and his collaborators[2] during the past seven years. The main results and the conclusions to which they lead are summarized below.

At the beginning of the work, it was felt that the most familiar case of aqueous solutions would be better postponed, since the complexity of the equilibria here undoubtedly confuses the issue. Generalizations for non-aqueous solutions were accordingly first sought, with the idea of testing their applicability to aqueous solutions (where data for the purpose are plentiful) at as early a stage as possible.

Ionization and Ideality.—The most significant characteristic of all conducting solutions is that they are non-ideal. Two non-associated substances, chemically inert one towards the other, never give a conducting solution on admixture. Ionization is consequently always accompanied by interactions between the "solvent" and "solute." How to connect the extent of these interactions with the character of the two components was the first problem to be attacked.

Now the character of the two components AB and CD in a conducting solution, as related to ionization, may obviously be varied by varying the "ionization tendency" (or "electro-affinity") of the constituent radicals A, B, C and D. A definite experimental criterion of such variations is afforded by electromotive force measurements (for inorganic radicals) and by comparison of ionization constants (for organic radicals). The results obtained in these fields have therefore been utilized to investigate the influence of the character of the constituent radicals upon interactions between the components in solutions of widely different types.

Compound Formation and Electroaffinity.—The first type of system studied contained but one variable radical. Dimethylpyrone (a weak base in aqueous solution) was chosen as one component, acids of widely divergent character were taken as the second, and the extent and stability of compound formation on admixture were established for each particular case by examination of the freezing-point depression curves. The only variable factor here was the electroaffinity of the radical X of the acid component HX. The experimental results demonstrated that compound formation in the system regularly increased as X became more negative.

Systems with more than one variable radical were next investigated. For example, aldehydes of widely divergent character were substituted for dimethylpyrone and tested against different acids. It was found that compound formation was uniformly more extensive the more electropositive the radical R of the aldehyde $R.CHO$ and the more electronegative the radical X of the acid HX. Similarly with systems of the type ester-acid the extent of compound formation increased regularly as the radicals R and R^1 of the ester $R.COOR^1$ were made more electropositive, or as the radical X of the acid HX was made more electronegative. With systems of the type acid-acid, finally, it was shown that the weaker acid HX functioned as a base with regard to the stronger acid HY, the stability of the resultant salt being greater the greater the difference in the electroaffinity of the radicals X and Y. Incidentally no fewer than 96 new organic compounds have been isolated in the course of this work.

Compound formation in binary organic mixtures is therefore fundamentally dependent upon diversity in character (i. e., differences in positive or negative natures of the constituent groups) of the two components.

Compound Formation and Ionization.—Before proceeding further, it was necessary to confirm a point which the above results had indicated as highly probable—namely, that the order of electroaffinity of different radicals in these organic mixtures was substantially the same as in aqueous solutions. This was satisfactorily established by careful conductivity determinations upon various systems of the types examined above. In a fixed ketone or ester, for example, different acids HX were found to give more highly conducting solutions the more electronegative the radical X. In a fixed acid HX, conversely, different ketones $R.CO.R^1$ or esters $R.COOR^1$ gave solutions of better conductivity the more electropositive the radicals R and R^1. Different acids HY in a fixed acid HX, finally, gave more highly conducting solutions the greater the difference in the electroaffinity of the radicals X and Y.

An important generalization may therefore be deduced: in a conducting solution, *compound formation* between the components and *ionization* proceed in parallel. Where compound formation is very slight, ionization is inappreciable; as compound formation increases in extent, ionization becomes evident; where compound formation is very marked, ionization is extensive.

The validity of this generalization for aqueous solutions may now be examined. Acids and bases obviously furnish a stringent test, since ionization here varies very considerably. Examination of the available data shows that *strong* acids invariably give stable hydrates, while among the myriads of *weak* acids listed in Beilstein not a single example of hydrate formation is indicated. Quantitative confirmation of the parallelism between hydrate formation and ionization for acids in aqueous solution was obtained from freezing-point depression determinations with a series of acids of gradually increasing strength. With bases the data, though not so extensive, showed similar agreement.

A Modified Ionization Theory.—On the basis of the above results, a modification of the current ionization theory has been proposed, under the assumption that "ionization is preceded by combination between solvent and solute and is, indeed, a consequence of such combination." According to this view, ionization in solutions is due not to solute alone nor to solvent alone, but to combination between the two to form unstable complexes. The various factors affecting the formation and stability of such complexes have been critically discussed elsewhere.[3] It has been shown that, while in simple molecules the two radicals are in general attracted to each other so strongly that no dissociation is observable, yet in complex "addition compounds" the attractive forces between the radicals are so diminished that disintegration of the complex into oppositely charged ions may readily occur. *Union with another molecule promotes ionization.*

The theoretical arguments in favor of this assumption and the experimental evidence in its support cannot be presented in detail here. The immediate consequence of the modified ionization theory, as connected with the anomaly of strong electrolytes, may however be indicated.

Instead of a simple equilibrium such as $KCl \rightleftharpoons K^+ + Cl^-$ for a uni-univalent electrolyte in aqueous solution, we now have to consider a complicated series of equilibria between many different molecular and ionic types, such as $(KCl)_x$; $(KCl)_x, (H_2O)_y$; $[(KCl)_x, Cl]^-$; $[(H_2O)_y, Cl]^-$, etc. To some of these equilibria the law of mass action alone may apply, to others the Milner-Ghosh conception of electrostatic equilibrium may need to be superadded. Little wonder, therefore, that the problem has baffled all attempts at explanation under the hypothesis $RX \rightleftharpoons R^+ + X^-$. If we persist in postulating the reaction as inconceivably simple, we can hardly expect to make progress, even in the field of dilute solutions, where many of the equilibria may be neglected.

It might seem that the complexity in the equilibria necessarily introduced by the consideration of so many molecular types could, after all, only make the problem still more intricate and elusive. The first effect, nevertheless, is in the opposite direction. Many subsidiary assumptions which have been attached to our theory of solutions in order to conceal the failure of the simple equation $RX \rightleftharpoons R^+ + X^-$ may now be shown to be unnecessary. For example, the hypothesis of the "catalytic activity of the undissociated molecule," which purports to explain why the speed of reactions such as ester catalysis is not exactly proportional to hydrogen ion concentration, may be discarded in favor of a view which recognizes several types of "hydrogen ion" (e. g., H^+, $[H(H_2O)]^+$, $[H(R.COOR^1)]^+$) each possessing a different catalytic activity. In the same way abnormal results in electromotive force measurements, which have been interpreted as demonstrating a variable activity of the hydrogen ion with concentration, may now be alternatively referred to variations in the ratios of the different types of "hydrogen ion" with concentration. There is no more reason, for instance, why the ion $[H(H_2O)_x]^+$ should show the same potential difference as the ion H^+ with respect to gaseous hydrogen than there is for the ion $[Cu(NH_3)_y]^{++}$ to exhibit the same potential difference as the ion Cu^{++} with respect to metallic copper. The extension of the theory of complex ions to simple aqueous solutions is evidently one of the first steps to be taken in the further development of the views here presented. Detailed discussion must be postponed to a later article.

Salts in Aqueous Solution.—The correlation of compound formation and ionization in this important field offers difficulties. Nearly all uni-

univalent salts are practically equally dissociated in aqueous solution, yet hydration values so far as we can judge from present data are widely divergent. It is a very striking fact, however, that with mercuric salts (the one series in which ionization varies considerably) *all* highly ionized salts yield hydrates (e. g., $Hg(NO_3)_2, 8H_2O$; $Hg(ClO_4)_2, 6H_2O$; $HgF_2,2H_2O$) while *all* slightly ionized are non-hydrated.

It appeared that the problem might be taken up most profitably in two stages—by the examination of systems of the type RX–HX (acid salts) and of the type RX–ROH (basic salts). By comparison of the results in these fields with those already obtained for systems of the types: HX–H_2O and ROH–H_2O it was hoped that generalizations for the more complex type RX–H_2O might be successfully formulated. The results obtained from a detailed investigation of two series of acid salts (formates and sulfates) have justified this procedure.

Acid Salts.—Compound formation in the system acid–salt is found to be directly dependent upon differences in electroaffinity, as before. With salts of highly electropositive metals (e. g., *K*, *Na*) extensive compound formation is obtained. As the metal approaches hydrogen in the electromotive series, compound formation decreases rapidly (with *Ba, Ca, Mg*) and finally becomes negligible (with *Ni, Fe, Cu*). With highly electronegative metals, however (*Hg*, Ag), compound formation once more becomes appreciable. Ionization again proceeds in parallel with compound formation, the best conducting solutions in pure formic or sulfuric acid being given by the salts of *K* and *Na*.

Solubility.—A new and very significant relationship which appears in these systems is that of solubility, which also parallels compound formation. The sulfates of *K* and *Na* are exceedingly soluble in pure H_2SO_4, those of *Ba, Ca* and *Mg* are decreasingly soluble, those of *Ni, Fe, Cu* practically insoluble. With electronegative metals the solubility again becomes appreciable (with *Hg*) and finally extensive (with *Ag*). The solubilities of the formates in formic acid follow exactly the same course.

That the rule is valid also in aqueous solutions may be demonstrated by applying it to systems of the type ROH–HOH. The hydroxides of the alkali metals are very soluble in water, those of the alkaline earths decreasingly soluble, those of *Ni, Fe, Cu* practically insoluble. Silver again shows an increase, limited however by precipitation of the more stable oxide.

In systems of the type RX–HOH, finally, the same rules hold if *R* is not very different in character from *H*, or if *X* is not very different in character from *OH*. Thus the salts of weak bases (e. g., ferric salts) or of weak acids (e. g., fluorides) show regular gradations of solubility in water, which substantially follow the generalizations given above. For salts of a strong acid with a strong base, however, where *R* and *X* both differ widely in character from H and OH, no such simple behavior is to be expected, nor is it found Rules for such systems have still to be discovered.

The reason for the above dependence of solubility upon compound formation becomes evident when we examine the freezing-point depression curve of a system such as Li_2SO_4–H_2SO_4. The absolute freezing-point of the salt is so high that, unless compound formation took place, solubility in the acid at ordinary temperatures would, according to the Schröder-Le Chatelier equation,[4] necessarily be negligible. The removal of simple solute molecules by combination with solvent, however, results in an increased solubility, and the extent of such increase depends, naturally, upon the extent of compound formation. The further development of this topic and the presentation of quantitative results will be given in forthcoming articles in the *Journal of the American Chemical Society*.

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[1] Kendall and Booge, *J. Amer. Chem. Soc.*, **39**, 1917 (2323).

[2] Carpenter, Ph.D. *Dissertation, Chicago,* 1915; Gibbons, *J. Amer. Chem. Soc.*, **37,** 1915 (149); Booge, Ph.D. *Dissertation,* Columbia, 1916; Gross, Ph.D. *Dissertation,* Columbia, 1919; Landon, Ph.D. *Dissertation,* Columbia, 1920; Adler, Ph.D. *Dissertation,* Columbia, 1920; Davidson, Ph.D. *Dissertation,* Columbia, 1920.

[3] Kendall and Booge, *loc. cit.;* Gross, *loc. cit.*

[4] Roozeboom, *Heterogene Gleichgewichte,* **2,** 1904 (270–84).

THE PELTER EFFECT

By Edwin H. Hall

Jefferson Physical Laboratory, Harvard University

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When a current of electricity goes from metal α, in which the free-electron fraction of the conductivity is $(k_f \div k)_\alpha$, to the metal β, in which the corresponding fraction is $(k_f \div k)_\beta$, ionization or reassociation must take place at the junction, according as $(k_f \div k)_\beta$ is greater or less than $(k_f \div k)_\alpha$. The resulting change of condition is the same whether we assume the ionization or reassociation to take place after or before the crossing of the interface between α and β. We shall use each of these hypotheses in turn, beginning with the first.

The gain of energy, and so the heat energy absorbed, by the unit quantity of electricity, 10 coulombs, in the form of electrons, in crossing the boundary from α to β is

$$\Pi = \left(\frac{k_f}{k}\right)_\alpha \left(P_\beta - P_\alpha\right)_f + \left(\frac{k_a}{k}\right)_\alpha \left(P_\beta - P_\alpha\right)_a + \left(P_\beta - P_\alpha\right) + \left[\left(\frac{k_f}{k}\right)_\beta - \left(\frac{k_f}{k}\right)_\alpha\right]\lambda_\beta, \qquad (1)$$

where $(P_\beta - P_\alpha)_f$ means the P_f of β minus the P of α, etc., the nomenclature according in general with that of my preceding papers.

From my paper in these Proceedings for April, 1918, we get, by integration of equation (5) there given,

$$(P_\beta - P_\alpha) = -(P_\beta - P_\alpha)_a. \qquad (2)$$

Integrating equation (6) of the same paper, we get

$$(P_\beta - P_\alpha) + (P_\beta - P_\alpha)_f = -\int_\alpha^\beta \frac{dp}{ne} = -\frac{RT}{e}\int_\alpha^\beta \frac{dn}{n} = \frac{RT}{e}\,.\log_\epsilon \frac{n_\alpha}{n_\beta}. \qquad (3)$$

From (1), (2) and (3) we get

$$\Pi = \left(\frac{k_f}{k}\right)_\alpha \frac{RT}{e}\,.\log_\epsilon \frac{n_\alpha}{n_\beta} + \left[\left(\frac{k_f}{k}\right)_\beta - \left(\frac{k_f}{k}\right)_\alpha\right]\lambda_\beta. \qquad (4)$$

If we had used at the beginning the hypothesis that the ionization occurred in α, just before the crossing, we should have obtained the expression

$$\Pi = \left(\frac{k_f}{k}\right)_\beta \frac{RT}{e}\,.\log_\epsilon \frac{n_\alpha}{n_\beta} + \left[\left(\frac{k_f}{k}\right)_\beta - \left(\frac{k_f}{k}\right)_\alpha\right]\lambda_\alpha. \qquad (5)$$

The total change of state being the same in the process which gives equation (5) as in that which gives equation (4), the total heat absorbed must be the same in both cases—that is, the Π of (5) is the same as the Π of (4).

Subtracting (5) from (4) we get

$$\left[\left(\frac{k_f}{k}\right)_\alpha - \left(\frac{k_f}{k}\right)_\beta\right]\frac{RT}{e}\,.\log_\epsilon \frac{n_\alpha}{n_\beta} + \left[\left(\frac{k_f}{k}\right)_\beta - \left(\frac{k_f}{k}\right)_\alpha\right]\left(\lambda_\beta - \lambda_\alpha\right) = 0, \quad (6)$$

whence

$$\lambda_\beta - \lambda_\alpha = \frac{RT}{e}\,.\log_\epsilon \frac{n_\alpha}{n_\beta}, \qquad (7)$$

or

$$\frac{n_\alpha}{n_\beta} = \epsilon^{\frac{(\lambda_\beta - \lambda_\alpha)e}{RT}}. \qquad (8)$$

Equation (8) has a familiar form, and in fact it might have been written down at once without circumlocution as soon as the fact was recognized, as it was in my paper of April, 1918, that the free electrons in the one metal are in equilibrium with those in the other, there being no circulatory motion of the electrons at the junction if the metals are at one temperature.

Making use of equation (7) we can change (4) to the form

$$\Pi = \left(\frac{k_f}{k}\right)_\alpha (\lambda_\beta - \lambda_\alpha) + \left[\left(\frac{k_f}{k}\right)_\beta - \left(\frac{k_f}{k}\right)_\alpha\right]\lambda_\beta, \qquad (9)$$

and so

$$\Pi = \left(\frac{k_f}{k}\right)_\beta \lambda_\beta - \left(\frac{k_f}{k}\right)_\alpha \lambda_\alpha. \qquad (10)$$

It is now in order to see how well values of Π calculated by means of this formula, with the tentative values of $(k_f \div k)$ and λ obtained for each metal in my paper on the Thomson Effect and Thermal Conduction,[1] agree with those found by Bridgman from his experiments. Close agreement is, of course, not to be expected, even if my theory is everywhere correct, for the values of $(k_f \div k)$ and λ given in the paper just mentioned were obtained without any regard whatever for the Peltier effect, they being merely certain values, not the only values, that will, approximately at least, meet the requirements of the Thomson effect and thermal conductivity. Moreover, the experimental data used in arriving at these values are doubtless faulty in many particulars; for example, in a number of cases I had no observed value of the thermal conductivity at 100° C., and was obliged to make use of a value calculated, not very safely, by means of the Wiedemann-Franz law.

Departing from the common practice, I have substituted bismuth for lead as the reference metal, taking as my definition of the Peltier heat the amount of heat (ergs) absorbed in the passage of $(1 \div e)$ electrons from any given metal to bismuth. All the values in the table below are to be multiplied by 10^4. *Con.* is constantan and *Man.* is manzanin.

TABLE 1

THE PELTIER HEAT

	OF VARIOUS METALS AGAINST BISMUTH								
	At 0° C.				At 100° C.				
	I Obs.	II Calc. 1	III Calc. 2	IV Calc. 3	V Obs.	VI Calc. 1	VII Calc. 2	VIII Calc. 3	IX V–VIII
Fe	247	798	311	251	310	861	323	226	84
Mo	219	814	327	217	319	911	373	252	67
Zn	211	822	335	212	274	904	366	225	49
Au	211	804	317	210	280	888	350	227	53
Cu	211	817	330	211	280	905	367	229	51
Ag	210	831	344	209	279	925	387	226	53
Tl	208	778	291	209	271	850	312	221	50
W	207	748	261	208	285	822	284	234	51
Sn	204	799	312	204	266	880	342	220	46
Mg	203	769	282	202	266	846	308	220	46
Al	202	772	285	201	263	844	306	210	53
Pt	195	783	296	194	245	847	309	195	50
Pd	188	747	260	188	232	795	257	182	50
Co	156	645	158	154	172	642	104	117	55
Ni	155	650	163	153	187	668	130	136	51
Bi	0	0	0	0	0	0	0	0	0
Con.	108	533	36	108	106	493	—45	55	51
Man.	207	591	104	206	275	604	66	223	52

Column I gives the observed, or experimentally found, values of Π, with respect to bismuth, at 0° C. Column II gives the values calculated, by means of equation (10), from the values of $(k_f \div k)$ and λ, for 0° C., given in my already printed paper on the Thomson Effect and Thermal Conduction. Comparison of these two columns shows that the calculated values are three or four times as great as the observed values, but this disparity, before any adjustments have been attempted, is not discouraging.

Column V gives the observed values for 100° C., while VI gives those found by means of equation (10) from the values of $(k_f \div k)$ and λ, for 100° C., given in the paper just mentioned. Here again the calculated values are some-fold larger than the observed values.

Columns III and VII, which are to be compared with I and V, respectively, are obtained from columns II and VI, respectively, merely by using new values of $(k_f \div k)$ and λ, for bismuth, these new values serving quite as well as the former ones, on the whole, for the Thomson effect and the thermal conductivity of bismuth.

Columns IV and VIII, which are to be compared with I and V, respectively, are obtained from III and VII, respectively, by using new values of $(k_f \div k)$ and λ for each of the other metals than bismuth, due regard being paid to the Thomson effect and the thermal conductivity in the selection of these new values.

The agreement between columns I and IV is as good as need be. The agreement between columns V and VIII is not so good, the values in V being in every case the larger, as column IX shows. A new readjustment in the case of bismuth, making $[(k_f \div k)\lambda]_{100}$ for this metal about 14% greater than the last re-adjustment left it, would reduce most of the differences shown in column IX to a negligible size. But such a re-adjustment cannot be made without introducing disagreements at other points. The disparities between columns V and VIII, together with equally serious disparities, at 100° C., between thermal conductivities given by my formulae and those found by other means, must stand for the present.

A plausible explanation of these discrepancies, where they are not to be acounted for by mere imperfection of experimental data, may be found in the crudity of my assumption that the number, n, of free electrons per cu. cm. of a metal, can be expressed by the formula

$$n = zT^q$$

where z and q are constants. In making this assumption I did not expect it to hold for so great a range of temperature as that between 0° and 100° C. In fact, I was greatly surprised to find, a year or so ago, that I could make the jump from 0° to 100° with any measure of success.

It is interesting to inquire how the theory of the Peltier effect set forth in this paper can deal with Bridgman's observations on the magnitude and sign of this effect between compressed and uncompressed pieces of the same metal. Testing twenty metals, including two alloys, he measured the heat absorbed by unit quantity of "positive electricity in passing from uncompressed metal to [the same] metal compressed" or, as I prefer to put it, by $(1 \div e)$ electrons in passing from the compressed to the uncompressed metal. Working from 1 atm. to 2000 kgm./cm.^{2} and from 0° C. to 100° C., he found this absorption to be positive throughout in fourteen of the metals; negative throughout in three, cobalt, magnesium, and manganin; mixed, sometimes positive and sometimes negative, in three, aluminium, iron, and tin.

In terms of my theory a positive effect here can be accounted for by a decrease of $(k_f \div k)\lambda$ under pressure, and a negative effect by an increase. It seems probable that $(k_f \div k)$ is generally decreased by pressure, and that the increase of $(k_f \div k)\lambda$ indicated for certain cases by Bridgman's experiments is to be attributed to an increase of λ sufficient to overbalance the decrease of $(k_f \div k)$. *A priori* one might expect λ to decrease with increase of pressure, causing reduction of volume, since, according to the formula $\lambda' = \lambda_c' + sRT$, it increases with rise of temperature, causing expansion. But it is unsafe to assume that a contraction caused by pressure will have the same effect on the properties of a substance as a contraction caused by fall of temperature. Thus Bridgman says:[2] "The volume of many metals at 0° C. and 12,000 kg. [per cm.^{2}] is less than the volume at atmospheric pressure at 0° Abs. The resistance of most metals tends towards zero at 0° Abs., but at 0° C. at the same volume the resistance is only a few per cent less than under normal conditions."

A change of about 8% in the value of $(k_f \div k)\lambda$ would account for the maximum Peltier effect between compressed and uncompressed bismuth, as observed by Bridgman, and a still smaller per cent change, in most cases much less than one per cent, would serve for the other metals dealt with in this paper.

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[1] These PROCEEDINGS, October, 1920, p. 613.

[2] *Proc. Amer. Acad. Arts & Sci.*, **52**, No. 9 (638).

ON THE ROOTS OF BESSEL'S FUNCTIONS

By J. H. McDONALD

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA

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The roots of the equation $J_n(z) = 0$ are known to be all real if $n < -1$. The methods of Sturm when applied to the function $J_n(z)$ show that the roots are increasing functions of n if $n > 0$, that is to say, denoting by $\psi_k(n)$ the k^{th} positive root $\psi_k(n') > \psi_k(n)$ if $n' > n > 0$. In the following it will be shown that $\psi_k(n') > \psi_k(n)$ if $n' > n > -1$ so that the inequality holds as well when $-1 < n < 0$.

Putting $f_n(z) = \sum \frac{z^r}{\Gamma(n+r+1)\Gamma(r+1)}$ so that $J_n(z) = \left(\frac{z}{2}\right)^n f_n\left(-\frac{z^2}{4}\right)$ the polynomials $g_\nu(z)$ are to be considered. They satisfy the recurrence formula $g_{\nu+1} = (n+\nu+1)g_\nu + zg_{\nu-1}$ with initial values $g_{-1} = 0$, $g_0 = 1$ and are connected with f_n by the relation $f_n = g_\nu f_{n+\nu} + zg_{\nu-1} f_{n+\nu+1}$. Putting $\Delta_\nu = g_\nu g'_{\nu+1} - g_{\nu+1} g'_\nu$ where differentiation is with respect to z it is known (Hurwitz, *Math. Ann.*, **33**, 1889, p. 246) that $\Delta_{\nu+2} = (n+\nu+2)g^2_{\nu+1} + z^2\Delta_\nu, \Delta_1 = n+1,\ \Delta_2 = (n+1)^2(n+2)$ and since $\frac{d}{dz}\left(\frac{g_{\nu+1}}{g_\nu}\right) = \frac{\Delta_\nu}{g_\nu{}^2}$ and $\Delta_1 = n+1$, it follows that $\frac{g_{\nu+1}}{g_\nu}$ is an increasing function of z if $n+1>0$.

If differentiation is taken with respect to n and $D_\nu = g_\nu \frac{dg_{\nu+1}}{dn} - g_{\nu+1}\frac{dg_\nu}{dn}$ it is found that $D_\nu = g_\nu{}^2 - zD_{\nu-1}$, $D_0 = 1$, $D_1 = (n+1)^2 - z$. For, from $\frac{dg_{\nu+1}}{dn} = g_\nu + (n+\nu+1)\frac{dg_\nu}{dn} + z\frac{dg_{\nu-1}}{dn}$ with use of the equation $g_{\nu+1} = (n+\nu+1)g_\nu + zg_{\nu-1}$, it follows that $g_\nu \frac{dg_{\nu+1}}{dn} - g_{\nu-1}\frac{dg_\nu}{dn} = g_\nu{}^2 - z\left(g_{\nu-1}\frac{dg_\nu}{dn} - g_\nu \frac{dg_{\nu-1}}{dn}\right)$. If a negative value is assigned to z $D_\nu>0$ and $\frac{g_{\nu+1}}{g_\nu}$ is an increasing function of n, with the condition as before $n+1>0$.

From these properties of $\frac{g_{\nu+1}}{g_\nu}$ it follows if z_1 is a root of $g_\nu = 0$ that $g_{\nu-1}(z_1)>0$ or <0 according as g_ν changes from negative to positive or the reverse when x increases through z_1. Denoting the dependence of g_ν on n by g_ν^n it follows if n' is slightly greater than n that $g_\nu^{n'}(z_1)>0$ in the first case and <0 in the second because in both cases $\frac{g_\nu^{n'}(z_1)}{g_{\nu-1}^{n'}(z_1)} > \frac{g_\nu^n(z_1)}{g_{\nu-1}^n(z_1)} = 0$. If z_1' denotes the root of $g_\nu^{n'}(z) = 0$ which differs slightly from z_1 it follows in both cases that $|z_1'|>|z_1|$. The roots of $f_n = 0$ are known to be the limits of the roots of $g_\nu^n(z) = 0$, $\nu = 1, 2, \ldots$, hence, if ρ_k denote the k^{th} root of $f_n = 0$ and $\rho_{k'}$ of $f_{n'} = 0$, $|\rho_{k'}| \geq |\rho_k|$. The equality can easily be excluded. This is the theorem stated at the beginning. It has been assumed that $z<0$ and if $n+1>0$, all the roots z_1 are <0 in fact.

A formula of another kind may be obtained as follows: Differentiating the equation $f_n = g_\nu f_{n+\nu} + zg_{\nu-1} f_{n+\nu+1}$ and equating the result to f_{n+1} expressed in terms of $f_{n+\nu+1}, f_{n+\nu+2}$, and comparing coefficients it is found that $g_\nu^{n+1} = g_\nu^n + (n+\nu+1)(g^\nu)' + (zg_{\nu-1}^n)'$ and $g_\nu^{n+1} - g_\nu^n = (g_{\nu+1}^n)'$.

An application of the second equation may be made to the calculation of the roots of $f_n = 0$. Assuming $n + 1 > 0$ as before, so that the roots of $g_\nu^n = 0$ are all real and negative, let z_1, z_2, z_3 be the k^{th} roots of $g_{\nu+1}^n$ g_ν^n and $g_\nu^{n+1} = 0$, then $z_1 > z_2 > z_3$ and the functions g all change in the same sense from negative to positive or the reverse when z passes through the corresponding root. If z' is the k^{th} root of $(g_{\nu+1}^n)' = 0$, from the above equation $g_\nu^{n+1}(z') = g_\nu^n(z')$ and it can be seen that $(g_{\nu+1}^n(z_2))' > 0$ or < 0, according as $g_{\nu+1}^n$ changes from negative to positive or the reverse when z passes through z_1. This allows a series of approximations to a root of $f_n = 0$, say the first one, as follows: Let z_2 be the root of $g_2 = 0$; form the quantities $z_3 = z_2 - \dfrac{g_3(z_2)}{g_3'(z_2)}$, $z_4 = z_3 - \dfrac{g_4(z_3)}{g_4'(z_3)}$, etc. These z's converge to the first root of $f_n = 0$. In the case of the first root the approximations are rational functions of n.

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PROCEEDINGS
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Volume 7 MARCH 15, 1921 Number 3

PEABODY MUSEUM ARIZONA EXPLORATION, 1920

By A. V. Kidder and S. J. Guernsey

Peabody Museum, Cambridge, Mass.

Communicated by W. H. Holmes, January 11, 1921

The season's field work allows us to delineate more clearly than has hitherto been possible the archaeological problems of the Kayenta district in northeastern Arizona. It seems best to give: first, a summary of our previous knowledge; second, an account of the new material brought to light; third, an outline of the present status of the investigation.

1. *Previous Knowledge of the Archaeology of the Region.*—The earliest culture was that of the Basket-makers, a dolicocephalic people who did not practice skull deformation, did not build permanent dwellings, had no pottery, did not grow cotton or beans or domesticate the turkey. They cultivated corn of a single simple variety. Typical arts were: the weaving of excellent coiled baskets, square-toed sandals, twined bags, fur-string robes. They used the spear-thrower and dart instead of the bow and arrow.

At a somewhat later period the country was inhabited by the Slab-house people (so named by the present authors). While their culture, known only from two or three badly preserved sites, was little understood, the following traits were recognized: the settlements were loose aggregations of small, round or oval, semi-subterranean rooms, with foundations of upright slabs and stone and adobe superstructures. The pottery was of two sorts: a well decorated black-on-white painted ware, with characteristic designs; and a rudimentary coiled ware with a few heavy, broad corrugations about the necks of the jars. Nothing else was known, not even such important features as skull-type, burial customs, textiles, basketry, sandals and agricultural products. Stratigraphic finds, however, had made the chronological position of the culture clear; it was earlier than the Cliff-dweller-Pueblo and later than the Basket-maker.

The third and latest group of remains was the Cliff-dweller-Pueblo. It was well known from the investigations of Cummings, Fewkes and the authors. Its leading traits were: houses of coursed masonry, the rooms rectangular and arranged in series to form, when the site permitted, definite pueblos; subterranean ceremonial rooms or kivas; crania with strong occipital deformation and apparently originally brachycephalic; cultivation of several varieties of corn, as well as of cotton and beans; domestication of the turkey and use of its plumage for feather-string cloth; use of the bow and arrow; manufacture of great quantities of pottery of distinctive types, black-on-white, red-and-yellow, red, and corrugated.

The interrelation of the above three groups was not understood. The Basket-maker was obviously the earliest, the Slab-house next, and the Cliff-dweller-Pueblo latest; but whether they were to be regarded as stages in the development of a single people, or whether they represented successive occupations of the region by two or even three different tribes, could not be decided on the basis of the evidence available.

2. The explorations of 1920 in Sagi Canyon have added many new data. A few Basket-maker remains were found in various caves; these ran entirely true to the previously established type and, as usual, showed no admixture of later material (Slab-house or Cliff-dweller-Pueblo). Furthermore, a hitherto unrecognized culture came to light. At the west end of a large cave were unearthed rooms with slab foundations and stone and adobe superstructures; in and about the rooms was rubbish which yielded the following material: rough undecorated pottery of distinctive shapes, mostly black, some gray, a very little red, no trace of coiling; many sandals elaborately decorated in relief work and in colors, with a broad scallop across the toe end of each; handsomely ornamented woven carrying straps; crudely made twined bags. There was no cotton or turkey-feather cloth. In front of the cave lay a small burial mound containing skeletons with long, undeformed crania and accompanied by pottery identical with that from the rooms.

In the same cave, but farther to the east were discovered burials of the Slab-house culture; the identification was made by means of typical Slab-house vessels deposited with the bodies. The individuals had strongly deformed skulls. With them were: a large decorated carrying basket of a type not found with Basket-maker burials; cotton cloth; turkey-feather cloth; a complete bow and part of another; a twilled yucca ring-basket; a carrying strap of Cliff-dweller-Pueblo type; and a piece of rush matting. No house-structures associable with these graves were found, but the site was only partially excavated.

At the eastern end of the cave was a group of typically Cliff-dweller-Pueblo rooms, containing an equally typical assortment of pottery, textiles, sandals and other objects of the Cliff-dweller-Pueblo period.

3. *Present Status of the Investigation.*—The Basket-maker culture is still to be considered the earliest in the region. A second culture, which we call the Post-Basket-maker, followed the Basket-maker. Its probable

descent from the Basket-maker is indicated by the dolicocephalic bead form and absence of cranial deformation; by the elaborateness of the sandal weaves; the presence of a degenerate type of twined-woven bag; and by the use of fur cloth instead of feather cloth. The absence of cotton from both cultures should be noted. Advances over the Basket-maker are seen in the appearance of pottery (albeit of a crude type), in the presence of permanent house-structures, and in the elaboration of the carrying straps.

The third culture is the one which we formerly called the Slab-house. As that term, based on a feature of the architecture, is equally applicable to the Post-Basket-maker we have discarded it and substitute the name Pre-Pueblo. This group is allied to the preceding one most closely apparently in house-types, and in the possession of pottery, though its characteristic wares are much the more highly developed of the two; it differs sharply from the Post-Basket-maker in the practice of skull deformation, in the possession of cotton, turkey-feather cloth and twilled basketry. Traits that it shares with the succeeding culture, the Cliff-dweller-Pueblo, are: skull deformation; decorated pottery; cotton; turkey-feather cloth; the bow.

From the above data it seems probable that the Basket-makers were the direct ancestors, both physically and culturally, of the Post-Basket-makers; the latter, however, had made considerable advances (houses, pottery). A direct line of descent from Post-Basket-maker to the Pre-Pueblo might be inferred from the similarity in house-types; but the Pre-Pueblo are in most respects much more nearly allied to their successors, the Cliff-dweller-Pueblo people, than they are to the earlier group.

To sum up: Basket-maker is probably ancestral to Post-Basket-maker; Pre-Pueblo to Cliff-dweller-Pueblo; the genetic relationship of Post-Basket-maker to Pre-Pueblo may be inferred, but is still doubtful. A more detailed knowledge of the material cultures of the two middle groups is necessary, as well as studies to determine whether or not skull deformation alone is capable of producing the marked appearance of brachycephaly exhibited by the crania of the two later groups.

THE OPEN MERCURY MANOMETER READ BY DISPLACEMENT INTERFEROMETRY[1]

By Carl Barus

Department of Physics, Brown University

Communicated February 2, 1921

1. *Apparatus.*—This is practically a U-tube, AmA', figure 1, with wide shanks, AA', connected by a channel, m, below. A and A' are cylindrical hollows, 2–3 cm. deep and about 5 cm. in diameter, cut in a rectangular

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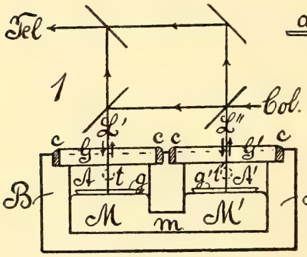
[1] Advance note from a report to the Carnegie Inst. of Washington, D. C.

block, BB', preferably of iron. The connection m must also be large in section, so as to admit of rapid flow from A to A'. The U-tube is charged with mercury MmM', M and M' being as shallow as possible to counteract the tendency to vibration. Thin plane parallel glass plates, gg', round discs of equal thickness and diameter, are floated on the mercury, which act as mirrors for the interferometer beams, L' and L'', and also materially check the tendency of the pool of mercury to vibrate. It would be desirable to be able to use the mercury surfaces at M and M' directly without the intervention of the plate; but within the city limits the fringes are unsteady and hard to find.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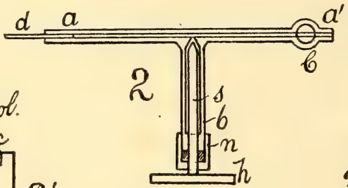
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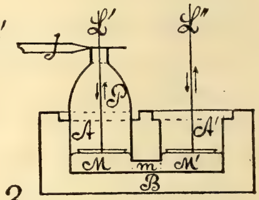
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The top of the iron block BB' is recessed as shown, to receive the plane parallel glass plates GG'. These like gg' must be equally thick; otherwise the fringes will be multiplied and faint. The annular space *cccc* between G and B is filled with resinous cement, poured in in the molten state. The air space AA' shut off in this way communicates with the atmosphere by two tubulures, t and t', in the front side.

The ray parallelogram of the quadratic interferometer of which $L'L''$ are the interfering rays should be vertical. The displacements of the achromatic fringes of white light are read off by a telescope with an ocular micrometer (scale part 0.01 cm.). The fringes parallel to the divisions of the micrometer are conveniently made a scale part in size. The block BB' should be mounted separately from the interferometer. If it is placed on the base of the latter, all manipulations there shake the mercury in BB' and it is necessary to wait for subsidence. This, however, occurs very soon, so that the separate mounting is not absolutely necessary. Without manual interference the fringes are about as quiet as in a solid apparatus.

2. *Experiments.*—To test this apparatus the air space AA' was left with a plenum of air. With A' communicating with the atmosphere, A was joined through t and a filimentary capillary glass or metal tube, to an apparatus by which slight pressure could be applied. In the first trials I attempted to use a water manometer controlled by a micrometer screw; but the vibrations of the meniscus were at once impressed on MM' so that the fringes were hard to keep at rest. I then devised the apparatus shown in figure 2, which is merely an adaptation of the pin valve of an oxygen tank, with a good micrometer screw, s, and stuffing box, n. The

head h of the screw is graduated. The barrel b is at right angles to the tube aa', which at a joins the capillary tube d, leading to t of figure 1. At the end a' there is a cock, C, which shuts off communication with the atmosphere. Thus when C is closed pressure is applied directly at A, figure 1, by rotating the head h in figure 2. This pressure is at once removed by opening C.

The apparatus worked surprisingly well. When C is closed and h rotated, the fringes may be placed anywhere in the field about as conveniently as with the micrometer screw at the mirror of the interferometer. There is, however, one difficulty which I have not thus far been able to remove. When the pressure increments exceed a certain small value, the plates gg' no longer rise and fall in parallel. The coincidence of images is destroyed and the fringes vanish. There is here a conflict with the capillary forces present at the edges of the disc. I endeavored to improve this by using small plates gg', anchored near the centre of MM' by 4 loose threads. But the advantage was not marked. Fringes a scale part in size will not be available for more than 50 scale parts, being sharpest in the middle. This is about half the diameter of field of the ordinary telescope. Curiously enough, fringes from the free surfaces of mercury vanish in the same way, probably owing to surface viscosity of the mercury, not absolutely pure.

3. *Equations and Pressure Observations.*—If the cock C, figure 2, is closed and the temperature for brief intervals is considered constant, Boyle's law may be written (ignoring signs of increments)

$$\frac{dv}{V} - \frac{dV}{V} = \frac{dp}{p} = \frac{dh}{76} \tag{1}$$

where V is the total volume enclosed, dv the increment at the micrometer screw hs, and dV the corresponding decrement equivalent to the pressure decrement dp. If a is the area of the piston at g $dV = a\,dh/2$ and if $V = aH$, H being the corrected air space at A, equation (1) becomes

$$\frac{dv}{V} - \frac{dh}{2H} = \frac{dh}{76}. \tag{2}$$

But dh on the interferometer is equivalent to n fringes of wave-length λ so that $dh = n\lambda/2$. Hence, finally

$$dv = V\frac{\lambda}{2}\left(\frac{1}{76} + \frac{1}{2H}\right).n. \tag{3}$$

This equation gives a test of the trustworthiness of the gauge.

In the apparatus used the following constants were found by measurement: $V = 66.8$ cm.^{3}, $a = 29.2$ cm.^{2}, $H = 2.29$ cm. The pitch of the screw was 0.073 cm. and its mean diameter 0.51 cm. Hence per turn $dv = 0.073 \times 0.204 = 0.0149$ cm.^{3} and $dv/V = 10^{-4} \times 2.23$ per turn. The mean wave-length being $\lambda = 6 \times 10^{-5}$ cm. equation 3 reduces to $n = 3.2$ fringes per turn of the screw hs.

In the experiments fringes of 1/2 scale part were installed. In separate experiments immediately after closing the cock C, a half turn of the screw produced a displacement of 8.3, 8.0, 8.0, 8.5, 8.5 scale parts, as the average, therefore 16.4 scale parts per turn or about 33 fringes per turn. This agrees as closely as may be expected with the number computed.

The pressure increment per turn of screw is $dp = n/2$ cm. of mercury or per turn of screw about 10^{-3} cm. Per fringe, therefore, 3×10^{-5} cm. of mercury as anticipated. A range of about 2 or 3 turns of screw was possible with each fringe, i. e., the range of pressure measurement should be from 3×10^{-5} to 3×10^{-3} cm. of mercury.

Experiments of the same kind were made in great variety. There is no difficulty in using much larger fringes so that 3×10^{-6} cm. of mercury should be appreciable. By exhausting both sides of the U-tube the apparatus becomes a vacuum gauge. I did not, however, attempt much work with it as the present apparatus was not well adapted for the purpose.

Air Thermometer.—If the cock C is permanently closed, the air space A becomes the bulb of an air thermometer of approximately constant volume. In this way the heat produced by the rays of light L' may be measured. In a variety of experiments of the kind, the mean result was about 10 scale parts or 20 fringes in a lapse of 210 seconds. If τ denotes absolute temperature, the intrinsic equation may now be written

$$\frac{dp}{p} + \frac{dv}{V} = \frac{d\tau}{\tau}$$

which reduces as above to

$$\frac{dh}{76} + \frac{dh}{2H} = \frac{d\tau}{\tau}.$$

Thus if $\tau = 300°$

$$d\tau = \tau n \frac{\lambda}{2}\left(\frac{1}{76} + \frac{1}{2H}\right) = 10^{-3} \times 2.1n° \text{ C.}$$

and for $n = 20$ in 210 seconds

$$d\tau = 0.042° \text{ C.}$$

or the heating produced was 2×10^{-4}° C. per second. Whether, supposing AA' to be filled with water, a pyrheliometer may be constructed on this principle I have yet to learn.

Other interesting results of the same kind might be mentioned. Thus if the screw stopcock, figure 2, is closed quickly (C being open) there is always a decided increment of pressure. In other words in consequence of the viscosity of air, the fine space at the plug is virtually a closure before the screw is checked by an actual closure.

If the glass plates G are removed and the air space A is closed by a pipe, P, tapering to a neck as in figure 3, it becomes a closed organ pipe adapted for the measurement of acoustic pressure. It may be blown by an adjustable embouchure, j, described heretofore (*Science*, **53**, 1921 (47)), or by resonance. Thus far, however, I have failed to find more than a pipe sounding under slightly increased or slightly diminished pressure.

Again if the apparatus, figure 3, is made of insulating material (*P*, *j* removed) and is provided with a disc electrode parallel and coaxial with the surface *M* used as a counter-electrode, the combination becomes an absolute electrometer of considerable interest. All the usual experiments of the electroscope may be performed by means of it, with the potential readings immediately in absolute units.

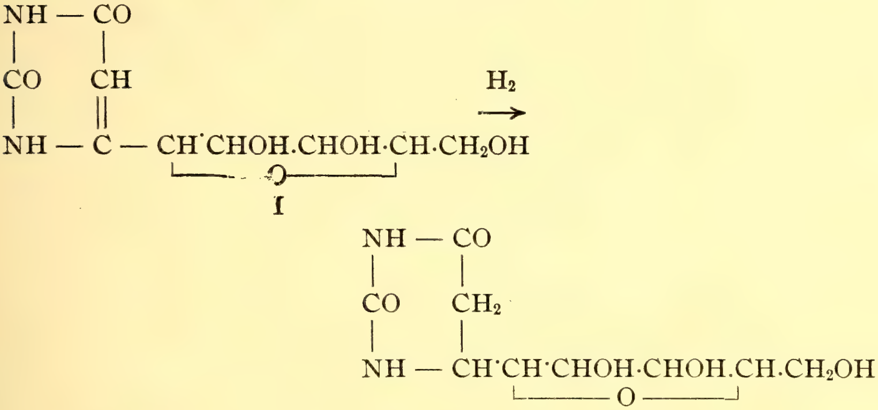
STUDIES ON CATALYSIS. I—THE REDUCTION OF URACIL TO HYDROURACIL

By Treat B. Johnson and Elmer B. Brown

Department of Chemistry, Yale University

Communicated February 1, 1921

Levene and LaForge,[1] in 1912, made the interesting observation that the nucleoside *uridine* I, which is obtained by hydrolysis of nucleic acid, is reduced practically quantitatively to dihydrouridine II by Paal's[2] method of catalytic reduction, namely, by means of colloidal palladium and hydrogen. This transformation involves an addition of hydrogen at the double bond joining positions 4 and 5 in the uracil nucleus and the change is represented as follows:



$$\begin{array}{l} \mathrm{NH - CO} \\ \;|\qquad\;| \\ \mathrm{CO\quad CH} \\ \;|\qquad\;\| \\ \mathrm{NH - C - \underbrace{CH{\cdot}CHOH.CHOH{\cdot}CH}_{O}{\cdot}CH_2OH} \end{array} \xrightarrow{H_2}$$

I

$$\begin{array}{l} \mathrm{NH - CO} \\ \;|\qquad\;| \\ \mathrm{CO\quad CH_2} \\ \;|\qquad\;| \\ \mathrm{NH - CH{\cdot}\underbrace{CH{\cdot}CHOH{\cdot}CHOH.CH}_{O}.CH_2OH} \end{array}$$

II

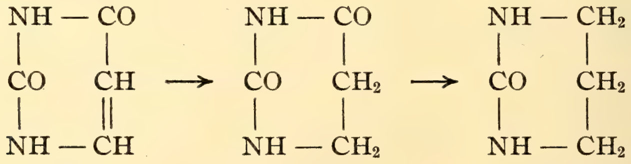
As far as the writers are aware, this is the first and only case described in the literature of the application of a catalytic process of reduction in the pyrimidine series. A striking fact revealed by the work of these investigators is the remarkable ease with which the sugar can be detached from this dihydrouridine molecule II by hydrolysis with acids. The corresponding uridine combination I is very stable and resistant to hydrolysis.

We have now inaugurated in the Sheffield Chemical Laboratory an extended research dealing with the application of catalytic methods of reduction and oxidation to pyrimidine, purine, hydantoin and glyoxaline compounds, and in this paper, the first of our series, we record the results obtained by direct reduction of the pyrimidine *uracil* with colloidal platinum.

The process which we have applied successfully for the reduction of uracil is based on the method of reduction developed by Skita.[3] This investigator used both colloidal palladium and platinum in his work and utilized as a support for his colloidal metal gum arabic. He showed also that his colloidal metals were active in both acid and alkaline solutions.

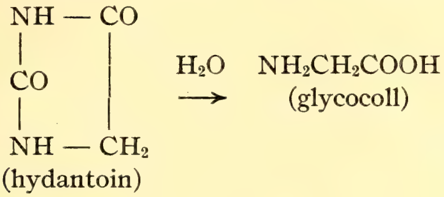
In our process we make use of a 10% solution of hydrochloroplatinic acid for preparation of our colloidal metal, and precipitate the activated metal in an aqueous solution of gum arabic by reduction with hydrogen gas without first introducing any colloidal metal to promote the reduction of the platinum chloride solution. Skita used in many cases a promoter to stimulate the formation of colloidal metal (Impfmethode). This was prepared by reducing platinum solution with formaldehyde in alkaline solution.

At a pressure of two atmospheres uracil III is reduced nearly quantitatively in aqueous solution to hydrouracil IV by action of hydrogen gas in presence of colloidal platinum. For example, when six grams of this pyrimidine were agitated in the reductor at a temperature of 75–85° C. and at a pressure of two atmospheres the reduction was complete at the end of seven hours, and almost a theoretical amount of hydrogen had added to the pyrimidine. The reduced pyrimidine melted at 272° C. and corresponded in all its chemical and physical properties with hydrouracil prepared by application of Hoffmann's reaction to succinimide.[4] It did not respond to Wheeler and Johnson's test for uracil.[5] We are now engaged in determining whether hydrouracil can be reduced still further by our catalytic method to the cyclic urea derivative of *syn*-trimethylenediamine V.

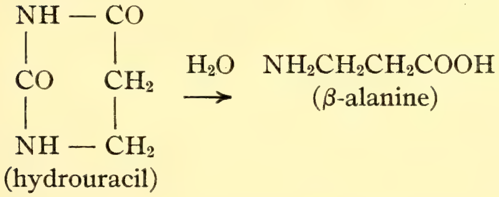


III IV V

From both a biochemical and a synthetical point of view the synthesis of hydrogenated uracils by direct reduction of the pyrimidine ring is a reaction of considerable importance. These reduced uracils bear a similar relationship to β-aminoacids as the hydantoins do to α-aminoacids. A practical method of reducing uracil combinations, therefore, opens up a new method of synthesizing representatives of this important class of aminoacids. The above relationships are expressed by the following equations:



(hydantoin) $\xrightarrow{H_2O}$ NH_2CH_2COOH (glycocoll)



(hydrouracil) $\xrightarrow{H_2O}$ $NH_2CH_2CH_2COOH$ (β-alanine)

Before undertaking our research on catalytic reductions we devised a practical machine whereby we might operate on a quantitative basis with large quantities of reagents. A description of this machine and a complete record of our preliminary work on uracil reduction will be published in a future number of the *Journal of the American Chemical Society.*

[1] *Ber.*, **45,** 1912 (619).
[2] *Ibid.*, **37,** 1904 (124); **38,** 1905 (1398, 1406).
[3] *Ibid.*, **45,** 1912 (3579, 3595).
[4] Lengfeld and Stieglitz, *Amer. Chem. J.*, **15,** 1893 (221).
[5] *J. Biol. Chem.*, **3,** 1907 (183).

THE APPLICATION OF A DIFFERENTIAL THERMOMETER IN EBULLIOSCOPY

By Alan W. C. Menzies and Sydney L. Wright

Department of Chemistry, Princeton University

Communicated by O. Veblen. Read before the Academy, November 17, 1920

For purposes of molecular weight determination of dissolved substances, the ebullioscopic procedure has certain advantages over the method of cryoscopy. Since, to use the latter, one must for convenience employ solvents whose freezing-points lie at easily accessible temperatures, it comes about that water, benzene and acetic acid have been commonly preferred. To obtain satisfactory values for the molecular weight of a solute, it is best to choose a solvent so alike to it chemically that compound formation between them shall be at a minimum. To be restricted in the choice of solvents is, therefore, a disadvantage. Boiling temperatures are preferable to freezing temperatures as disfavoring the formation of exothermic compounds. A majority of organic compounds boil normally below 300° C. If, as is true, the solvent must differ by at least 150 degrees in boiling-point from the solute in order that the volatility of the latter shall not vitiate the results, then the use is indicated of such solvents as boil below about 100°, and these solvents are indeed those most commonly used. The low solubility of certain solutes at the freezing-points of the available solvents is a factor against cryoscopy. On the other hand, the precision attainable in cryoscopy has, in the past, been considerably greater than that realized in ebullioscopy, even with apparatus fortified against error by the superimposition of a multiplicity of paraphernalia.

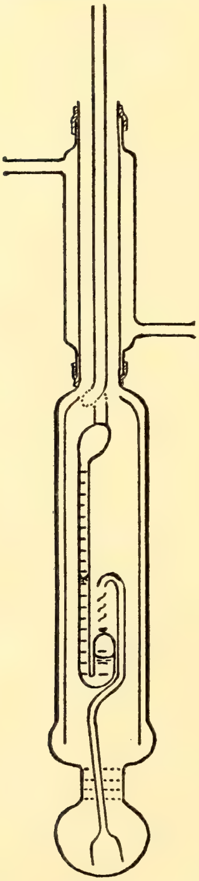


FIG. 1

For these and other reasons, a very great deal of work has been carried out by very many investigators with a view to improving the ebullioscopic technique. One may but mention the two dozen papers by Beckmann and his collaborators;[1] and the work of Hite,[2] Orndorff and Cameron,[3] Jones,[4] Ross Innes,[5] Meyer and Desamari,[6] Drucker,[7] Sakurai,[8] Landsberger,[9] Walker and Lumsden,[10] McCoy,[12] Smits,[13] Rijber,[14] Ludlam,[15] Erdmann and Unruh,[16] Lehner,[17] Turner.[18] Even a single publication by one writer—Hite—reports the trial of over 100 different forms of apparatus. In all these, and many other investigations, the bulb of a mercurial thermometer has been submerged beneath the surface of the boiling solution, and much of the work has been directed to the avoidance of the consequent irregular superheating. In our opinion, the two most successful improvements made since 1892 have originated in this country, the introduction, namely, by Bigelow[19] of electric heating, and by Cottrell[20] of a "lift-pump" which pumps the solution over the bulb of the thermometer, which no longer need be submerged in the solution, but is located above the solution in the vapor phase. When Cottrell's device is employed, there remain outstanding the errors inherent in the Beckmann type of thermometer, the error due to uncertainty as to actual concentration, and the very serious error due to change of boiling-point caused by change of barometric pressure during the observations. This last may, in part at least, be overcome by the employment of somewhat cumbrous "manostats,"[21] by especially constructed aneroid barometers,[6] or by the use of a second, duplicate, apparatus[22] operated simultaneously and containing another Beckmann thermometer immersed in the pure boiling solvent.

In 1910, Menzies[23] had described a simple apparatus, independent of barometric change, which gives molecular weights of solutes at their boiling-points by direct measurement of lowering of vapor pressure, without the use of any thermometer. Continuing this effort at simplification, he found that the application of the differential thermometer of the preceding article to ebullioscopy with the Beckmann type of apparatus served only to render more glaring the inevitable irregularities inseparable from that method of procedure. When, however, Cottrell's paper appeared in 1919, the first opportunity was taken to secure collaboration in working out the simplest possible apparatus and technique which should free ebullioscopy from all its most serious drawbacks.

The diagram, fig. 1, shows a boiling tube, whose narrowed upper portion itself serves as condenser tube. A glass cylinder, open above and below and of diameter slightly less than the boiling tube itself, is located concentrically within the latter, and confers the advantages of a double jacket. Within this is supported the differential thermometer, on the lower bulb of which hangs the pump, loosely, as a hat on a peg.[24] The absence of rigid connection diminishes breakage, besides adding flexibility, literal and metaphorical. The boiling tube is heated by a common Bunsen burner, with a low flame protected by the customary metal draught-shield. When boiling begins, portions of the liquid in the bulb are carried by the vapor in fragments up the pump and discharged to form a thin film over the lower bulb of the thermometer, which thus attains the temperature at which this liquid, or solution, is in equilibrium with the vapor. The upper bulb acquires the temperature of the pure boiling solvent. The thermometer registers the difference between these two temperatures. Equilibrium is reached so rapidly that fresh additions of solute may be made as fast as they can be weighed out. Contrary to all ebullioscopic tradition, the apparatus will give good results in a room that is not free from draughts. Unlike every other ebullioscopic apparatus, this form is entirely free from all corks, ground glass joints or stoppers.

The sources of error, especially that due to barometric change, eliminated by the use of the new type of differential thermometer have been briefly referred to in the preceding article. The advantages gained by removing the lower bulb from immersion in the boiling liquid have been well explained by Cottrell[20] and confirmed by Washburn and Read.[25] The chief residual source of error lies in the uncertainty as to the true concentration of the boiling solution as deduced, not by subsequent analysis, but simply from the weights of solvent and solute used. Part of the solvent is absent from the solution in the form of condensate on the walls, and part is absent in the form of vapor. This source of error, ever present in the Beckmann type of apparatus, has often failed of due consideration. To minimize it, we constrict the boiling tube above the lower bulb, mark cc. graduations on the cylindrical neck so developed, and read off the actual working volume of the solvent an instant after causing ebullition to cease, while the tube is still wet with condensate and filled with vapor.

A simple water-filled differential thermometer suffices for the range of temperature from the boiling-point of water to that of ether, thus embracing the use of such other solvents as carbon bisulphide; acetone; chloroform; methyl, ethyl and propyl alcohols; ethyl formate, acetate and iodide; carbon tetrachloride and benzene; and allowing a choice adequate for ordinary purposes. No "setting" of the thermometer for the use of different liquids is required.

Details of procedure and of results will be given in more complete publications elsewhere. Different sets of determinations with the same materials, carried out on different days, yield results concordant within one-half of one per cent. In spite of the simplicity of the apparatus, we venture to think that this form of the ebullioscopic method compares favorably with the customary form of the cryoscopic method in ease and speed of operation as well as in precision of results.

[1] References to the numerous papers of Beckmann and his collaborators may be found in the work of Jellinek, C., *Stuttgart, Lehrbuch der Physikalischen Chemie,* **2,** 1915 (783 et seq.).

[2] Hite, *Baltimore, Md., Amer. Chem. J.,* **17,** 1895 (514).

[3] Orndorff and Cameron, *Ibid.,* **17,** 1895 (517).

[4] Jones, H. C., *Easton, Pa., The Freezing-Point, Boiling-Point and Conductivity Methods,* 1912.

[5] Innes, R., *London, J. Chem. Soc.,* **81,** 1902 (682).

[6] Meyer and Desamari, *Berlin, Ber. deut. Chem. Ges.,* **42,** 1909 (797).

[7] Drucker, *Leipzig, Z. phys. Chem.,* **74,** 1910 (612).

[8] Sakurai, *London, J. Chem. Soc.,* **61,** 1892 (989).

[9] Landsberger, *Berlin, Ber. deut. Chem. Ges.,* **31,** 1898 (461).

[10] Walker and Lumsden, *London, J. Chem. Soc.,* **73,** 1898 (502).

[12] McCoy, H., *Baltimore, Md., Amer. Chem. J.,* **23,** 1900 (502).

[13] Smits, A., *Amsterdam, Proc. Akad. Wetens.,* **3,** 1900 (86).

[14] Rijber, *Berlin, Ber. deut. Chem. Ges.,* **34,** 901 (1060).

[15] Ludlam, *London, J. Chem. Soc.,* **81,** 1902 (1193).

[16] Erdmann and Unruh, *Hamburg, Z. anorg. Chem.,* **32,** 1902 (413).

[17] Lehner, *Berlin, Ber. deut. Chem. Ges.,* **36,** 1903 (1104).

[18] Turner, *London, J. Chem. Soc.,* **97,** 1910 (1184).

[19] Bigelow, *Baltimore, Md., Amer. Chem. J.,* **22,** 1899 (280).

[20] Cottrell, *Easton, Pa., J. Amer. Chem. Soc.,* **41,** 1919 (721).

[21] Smits, A., *Leipzig, Z. phys. Chem.,* **39,** 1902 (415); Burt, *London, J. Chem. Soc.,* **85,** 1904 (339); Drucker, *loc. cit.;* Beckmann, *Leipzig, Z. phys. Chem.,* **79,** 1912 (565); etc.

[22] Beckmann, *loc. cit.;* Washburn and Read, *Easton, Pa., J. Amer. Chem. Soc.,* **41,** 1919 (729); Sluiter, *Amsterdam, Proc. Akad. Wet.,* **17,** 1914 (1043).

[23] Menzies, A. W. C., *Easton, Pa., J. Amer. Chem. Soc.,* **32,** 1910 (1615).

[24] The complete apparatus, with both boiling-tube and water-filled thermometer of Pyrex glass, is furnished by Messrs. Eimer and Amend, 205 Third Avenue, New York, N. Y.

[25] Washburn and Read, *vide* note 22, *supra.*

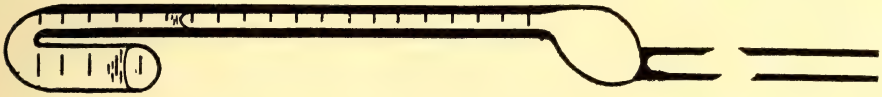
A DIFFERENTIAL THERMOMETER

BY ALAN W. C. MENZIES

DEPARTMENT OF CHEMISTRY, PRINCETON UNIVERSITY

Communicated by Oswald Veblen, January 20, 1921

One type of differential thermometer measures the difference in temperature existing at the same location at different times; a second type measures the difference in temperature existing simultaneously at different points in space. The thermometer here described is of the latter type.



A well-known differential thermoscope of this type consists of two glass bulbs containing air, otherwise closed but communicating with each other through a U-tube partly filled with oil, whose change of level indicates change of temperature by responding to change of gas pressure within the bulbs. When this instrument is developed into a differential thermometer, certain disadvantages become apparent, of which three will here be mentioned. (1) If the manometric liquid is caused to run into one of the bulbs by accidental tilting, perhaps during transportation, then it is difficult to return the liquid into precisely the same position as it occupied when the instrument was scaled. If stopcocks are introduced in the effort to avoid this inconvenience, the cure may become worse than the disease, because of zero-creep. (2) In the presence of permanent gas, the manometric liquid becomes, in practice, not infrequently broken into threads, separated by short columns of the gas. (3) Although oils furnish very sensitive manometric liquids, their use, or, indeed, the use of any liquid other than the insensitive mercury, allows the entrance of an error that has been too little appreciated. The incidence of this error in tensimetric work has been pointed out by the writer in another connection.[1] The error in question is caused by the fact that a gas at higher pressure has a larger weight solubility than the same gas at a lower pressure. The permanent gas, always slightly soluble in manometric liquids other than mercury, therefore passes by a process of solution and diffusion from the side of higher to that of lower pressure. For this reason even stopcock-free instruments of the kind referred to suffer from slow zero-creep.

In order to avoid these and other disadvantages, all that is necessary is to abandon entirely the use of permanent gas. One selects as manometric fluid not oil but some liquid whose change of vapor pressure per degree in the range of temperature where the differential measurements are to be made is such as to cause differences of vapor pressure in the two bulbs of the thermometer that will register themselves by adequate differences of level in the manometer. The diagram, figure 1, shows one simple form useful in ebullioscopy, made from glass tubing a few mm. in bore and having, without its handle, a length of perhaps 12 cm. Permanent gas is removed prior to sealing by the process of boiling out familiar to many who have had occasion to measure vapor pressures. For reading the difference of level of the two liquid surfaces, a mm. scale may be etched on both limbs. For many purposes, water is a suitable filling liquid; but it is obvious that the sensitiveness in any particular range of temperature may be given widely different values according to the rate of change of vapor pressure and the density of the liquid selected. The change of temperature between upper and lower bulbs that will cause a change of level of, say, 1 mm. in the height of the column of the filling liquid may be computed in an obvious manner from the known vapor pressures and densities of this liquid, and the results tabulated with temperature as argument. Thus, for water, at 57°, 80°, and 100°, these values are 0.01180°, 0.004969°, and 0.002599°, respectively.

Although the thermometers are of different general types, as indicated above, it is perhaps of interest to compare this differential thermometer with the Cavendish-Walferdin[2] metastatic type as elaborated by Beckmann,[3] and as applied in the same field, for example that of ebullioscopy. With regard to length of scale per degree, the Beckmann mercurial type is limited by the usable size of bulb, and by the permissible narrowness of the capillary, so that a centrigrade degree corresponds customarily to 40 or 50 mm. movement of the mercurial thread. The type here described is not thus limited in this respect, for a very low-boiling liquid may be used for filling. With the simple water filling, however, one degree centigrade at 80°, a temperature close to the boiling-points of the two favorite solvents benzene and ethyl alcohol, corresponds to an observed change of length of over 200 mm. In both types the length of degree varies with the actual temperature. As to range, the Beckmann type is restricted to that between $-39°$ and $+250°$ C.[4] While this is a much larger range than can be conveniently covered by the use of a single chosen liquid in the newer type, the simplicity of construction makes possible such a wide choice of filling liquids that a much wider range is easily available in the direction of lower as well as of higher temperatures. In comparing precision, one has to bear in mind, for the Beckmann type, possibilities of error due to (1) lack of uniformity of bore, (2) hysteresis in change of volume of bulb, (3) effect of pressure on volume of bulb, (4) sticking of mercury in capillary, (5) exposed thread, (6) difference of radiation to and from bulb, (7) departure of apparent degree from true degree. For the type here described, no one of the first six of these sources of error is important, for reasons that will be sufficiently obvious. In regard to (6), it may perhaps be said that the change with environment of radiation loss suffered by a Beckmann thermometer at temperatures far from room temperature is here largely eliminated because suffered alike by upper and lower bulb. With regard to (7), it is indeed most necessary to employ a factor, different for each temperature, to convert observed readings to temperature; and this factor may be criticized as inconvenient to use, inaccurate in value and laborious of computation. But the use of a similar factor is likewise necessary, albeit frequently neglected, in the case of the Beckmann thermometer, whose degree, if true at 0°, is, for example, about 3% in error at 80°.[5] The accuracy of the conversion factor for the newer type is dependent in part on the accuracy with which the vapor pressure of the filling liquid is known. For such liquids as would be employed, and within the ranges of temperature that come here in question, this quantity, the vapor pressure, can now be measured to better than one part in one thousand.[6] The table that will be published for the filling liquid water, in a more comprehensive article elsewhere, was derived by differentiation from the highly satisfactory, if cumbrous, equation connecting temperature and vapor pressure of water given in the 1918 edition of the Smithsonian Meteorological Tables, and has an accuracy of just this order. The process of computing factors for, perhaps, each tenth part of a degree over a considerable range of temperature may indeed be laborious; but, once published, the factor table may be used by everyone. This inconvenience, therefore, is shifted from the shoulders of the user of the thermometer to those of him who first computed the table. It may be added that the Beckmann type is considerably more cumbrous as well as very much more fragile than the type here described, which one constructs from stout-walled Pyrex tubing.

In certain respects, therefore, it would appear that this type of differential thermometer has advantages over the Walferdin metastatic type as elaborated by Beckmann; and the question arises as to whether such other factors as are peculiar to a given application are favorable to its use. In studying its application in ebullioscopy, for example, as outlined in the article following, one finds that the important disturbing factor, peculiar to ebullioscopy in its incidence, of barometric fluctuation does not measurably affect the readings of the newer type, while such pressure fluctuations are one of the chief outstanding sources of error when the metastatic type is used. Another application in a different field may be described in the near future.

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[1] Menzies, A. W. C., *Easton, Pa., J. Amer. Chem. Soc.*, **42,** 1920 (1951–1956).

[2] Cavendish, *London, Phil. Trans. Roy. Soc.*, **50,** 1757 (300); Walferdin, M., *Bull. Soc. geol. de France*, **13,** 1841–2 (113).

[3] Beckmann, E., *Leipzig, Zeit. physik. Chem.*, **2,** 1888 (644); **51,** 1905 (329).

[4] Cf. Staehler, A., *Leipzig, Arbeitsmethoden in der Anorg. Chemie,* **3,** i, 1913 (106).

[5] Staehler, A., *Ibid.*, p. 108.

[6] Cf. Smith, A., and Menzies, A. W. C., *Easton, Pa., J. Amer. Chem. Soc.*, **32,** 1910 (1412–1434).

NORMALIZED GEOMETRIC SYSTEMS

By Albert A. Bennett

University of Texas

Communicated by E. H. Moore, January 18, 1921

The notion of norm or numerical value of a complex quantity, $c = a + b\sqrt{-1}$, namely, $|c| = \sqrt{a^2 + b^2}$, as it arises in algebra, has a more or less immediate generalization to more extensive matric systems. The three important properties: (1) $|c_1 + c_2| \leq |c_1| + |c_2|$; (2) $|c_1 . c_2| = |c_1| . |c_2|$; (3) $c|$ is the positive square root of a positive definite quadratic form, are carried over at the expense only of replacing (2) by (2′) $|c_1 . c_2| \leq |c_1| . |c_2|$ and allowing in place of (3), (3′) $|c|$ is the positive square root of a positive definite form, Hermitian or quadratic. By $c_1 . c_2$ in these geometrical examples is meant the inner product[1] or a generalization of it. Two other generalizations of norm have been of great importance. The first of these is that of the theory of algebraic numbers,[2] where (1) is dropped, (2) is retained, and in place of (3) one has, Norm of c is a certain function of the nth degree, n being the order of the algebraic field. The second is that of a general theory of sets as treated for example by Fréchet,[3] where (1) is retained, (2) and (3) are dropped. The theory of integral equations as usually developed is geometrical in an infinity of dimensions and retains (1), (2′), (3′). It is noted that instances in which (1) and (2′) are retained usually keep (3′) also. Now the importance of (3′) is chiefly that it implies (1) and (2′) with the conventions as to linearity and so forth usually assumed. The converse that (1) and (2′) imply (3′) is false. It is of interest to show that most of the familiar properties of the norm may be retained, in particular (1) and (2′), when the norm is positive definite but otherwise largely arbitrary.

Three discussions bearing on this topic may be referred to. First, a geometrical study involving points but not their duals, by Minkowski, in his *Geometrie der Zahlen.*[4] The great generality of the idea of norm is there beautifully developed although it is not carried so far as it is here; but since the concept of the point dual is not brought in by Minkowski, most of the ideas here discussed are not found there. A second discussion, involving inner products, but treating only a very special case of the non-quadratic norm for an infinite number of variables is given by F. Riesz[5] in examining the convergence of bilinear forms. The third discussion involving only a scalar system, and hence without inner products, between elements of different systems is given by Kürschàk.[6] It is perhaps the most suggestive system of scalars in the literature in which $\|n\|$ may be less than n, for n a natural number.

The following treatment relates under one head the notions of convex region,[4] the triangle property,[7] the linearly homogeneous property of distance[8] or norm, conjugate norms,[5] the inner product,[1] convergence of a bilinear form,[5] Minkowski's gauge form[4] and Schwarz's inequality[9] (so-called), as these occur in geometry, hypercomplex number theory, integral equations, and more generally, general linear analysis.[10]

The following theorem may be proved: *Every geometric system admitting a gauge set may be normalized by means of that set.* In order that there may be no ambiguity an extended sequence of definitions will be given to cover all terms used.

If P is a proposition concerning a system, S, it may be that another system, T, is such that the proposition P has a meaning for the system T. The content of the proposition may be seriously altered while its form is not affected except in the sense that S is replaced by T. We may employ the functional notations $P(S)$ and $P(T)$ to suggest that the form of the proposition is carried over unaltered from one system to the other. To avoid repetition certain propositional functions will be here listed for reference. These serve as definitions, whenever all of the terms appearing in a proposition have been themselves defined, for example, "*Reg* (L)," below, acquires a meaning only in a system in which 0_L and 1_L are defined.

The "Addition Proposition,"[11] *Add.* (R): (*i*) R is a system comprising elements, r, and a rule of binary combination, $+$. (*ii*) For any r_1, r_2, of R, $r_1 + r_2$ is a uniquely defined element of R. (*iii*) For any r_1, r_2, of R, $r_1 + r_2 = r_2 + r_1$. (*iv*) For any r_1, r_2, r_3, of R, $r_1 + (r_2 + r_3) = (r_1 + r_2) + r_3$. (*v*) There is an element O_R of R (also denoted by merely O), such that for every r of R, $r + O_R = r$. (*vi*) For a given r_1 of R, there is not more than one element r of R, such that $r_1 + r = r_1$. (*vii*) For a given r_1 of R, there is one and only element, r, of R such that $r_1 + r = O_R$—this element, r, is denoted by $-r_1$.

The "Multiplication Proposition," *Mult.* $(R, S, T. P)$: (*i*) There are systems R, S, T, P, comprising elements, respectively, r, s, t, p, and two rules of binary combination $+$ and $.$, and such that with respect to $+$, the addition proposition is valid for each of the systems. (*ii*) For any r of R, s of S, t of T, $r.s$ and $s.t$ are uniquely defined, and $s.t$ is an element of P. (*iii*) For any r_1, r_2 of R, and s_1, s_2 of S, it is true that $r_1.(s_1 + s_2) = r_1.s_1 + r_1.s_2$ and $(r_1 + r_2).s_1 = r_1.s_1 + r_2.s_1$. (*iv*) For any r of R, s of S, t of T, it is true that $r.(s.t) = (r.s).t$. (*v*) For any r of R, and s of S, it is true that $O_R.s = O_T$, and $r.O_S = O_T$.

The "Commutative Proposition," *Com.* $(C.M.L.)$: (*i*) C is a subsystem of M. (*ii*) For each c of C and l of L, $c.l = l.c$. (*iii*) There exists an element 1_C, of C (also denoted merely by 1) such that for every l, $1_C.l = l$.

The "Regular Proposition," *Reg.* (L): (*i*) The elements of L fall into two mutually exclusive subsets L_{o*} and L_∞. (*ii*) The elements of L_{o*} fall into two mutually exclusive subsets, L_o and L_*. (*iii*) O_L is an element of L_o. (*iv*) 1_L if it has been defined and exists, is an element of L_*. (*v*) For every l of L, $-l$ is in the same set as l. (*vi*) There exists at least one l_* other than 1_L .

The "Regular Multiplication Proposition," *R. Mult.* (R, S, T, P): (*i*) Reg. (R), Reg. (S), Reg. (T), Reg. (P). (*ii*) Mult. (R, S, T, P). (*iii*) Mult. $(R_{0*}, S_{0*}, T_{0*}, P_{0*})$. (*iv*) Mult. (R_0, S_0, T_0, P_0).

The "Regular Commutative Proposition," *R. Com.* (C, M, L): (*i*) Com. (C, M, L). (*ii*) For every c of C and l of L, $c_0.l_0$, $c_0.l_*$ and $c_*.l_0$ are of L_0; $c_*.l_*$ is of L_*; $c_\infty.l_*$, $c_*.l_\infty$, and $c_\infty.l_\infty$ are of L_∞. (*iii*) 1_C is of L_*. (*iv*) For every l_1 and l_2 of L, if neither l_1 nor l_2 is in L_∞, $l_1 + l_2$ is not in L_∞, and if both l_1 and l_2 are in L_0, $l_1 + l_2$ is in L_0.

The "Regular Conjugate Proposition," *R. Conj.* (C, R, S): For each r of R, there is a c of C and an s of S, such that (*i*) $r.s = s.r = c.c$; (*ii*) if r is of R_0, c is of C_0, s is of S_0; if r is of R_*, c is of C_*, s is of S_*; if r is of R_∞, c is of C_∞, s is of S_∞.

The "Normal Proposition," *Nrm.* (L): (*i*) Each l of L has associated with it a unique value, $\|l\|$. (*ii*) Reg. (L). (*iii*) For each l_0, $\|l_0\| = 0$; for each l_*, $\|l_*\|$ is a positive real finite number; for each l_∞, $\|l_\infty\|$ is positively infinite. (*iv*) For each l of L, $\|-l\| = \|l\|$. (*v*) If 1_L exists, $\|l_L\| = 1$. (*vi*) There exists at least one l_* for which $\|l_*\|$ is different from unity.

The "Normal Multiplication Proposition," *N. Mult.* (R, S, T, P): (*i*) Nrm. (R), Nrm. (S), Nrm. (T), Nrm. (P). (*ii*) Mult. (R, S, T, P). (*iii*) $\|s.t\| \leq \|s\|.\|t\|$.

The "Normal Commutative Proposition," *N. Com.* (C, M, L): (*i*) Com. (C, M, L). (*ii*) For every c of C and l of L, for which $\|c\|.\|l\|$ exists, $\|c.l\| = \|c\|.\|l\|$. (*iii*) $\|1_C\| = 1$. (*iv*) For every l_1 and l_2 of L, $\|l_1 + l_2\| \leq \|l_1\| + \|l_2\|$.

The "Normal Conjugate Proposition," *N. Conj.* (C, R, S): For each r of R, there is a c of C and an s of S, such that (*i*) $r.s = s.r = c.c$, (*ii*) $\|r\| = \|c\| = \|s\|$.

The "Linear Proposition," *Lin.* (C, M, L): (*i*) Mult. (M, M, L, L). (*ii*) Mult. (M, L, M, L). (*iii*) Com. (C, M, L).

In the above, L is called a *linear system*, with M as *system of multipliers*, and C as *commutative subsystem* of M.

The "Hypernumber Proposition," *Hyp.* (C, H): (*i*) Lin. (C, C, C). (*ii*) Lin. (C, H, H).

In the above, H is called a *system of hypernumbers* with C as commutative subsystem.

The "Vector Proposition," *Vect.* (C, H, V): (*i*) Hyp. (C, H). (*ii*) Lin. (C, H, V).

In the above, V is called a *system of vectors*, with H as *associated system of hypernumbers*.

The "Regular Linear Proposition," *R. Lin.* (C, M, L): (*i*) R. Mult. (M, M, L, L). (*ii*) R. Mult. (M, L, M, L). (*iii*) R. Com. (C, M, L).

The "Regular Hypernumber Proposition," *R. Hyp.* (C, H): (*i*) R. Lin. (C, C, C). (*ii*) R. Lin. (C, H, H).

The "Regular Vector Proposition," *R. Vect.* (C, H, V): (*i*) R. Hyp. (C, H). (*ii*) R. Lin. (C, H, V).

The "Normal Linear Proposition," *N. Lin.* (C, M, L): (*i*) N. Mult. (M, M, L, L). (*ii*) N. Mult. (M, L, M, L). (*iii*) N. Com. (C, M, L).

The "Normal Hypernumber Proposition," *N. Hyp.* (C, H): (*i*) N. Lin. (C, C, C). (*ii*) N. Lin. (C, H, H).

The "Normal Vector Proposition," *N. Vect.* (C, H, V): (*i*) N. Hyp. (C, H). (*ii*) N. Lin. (C, H, V).

The "Geometric Proposition," *Geom.* (C, H, X, U): (*i*) Vect. (C, H, X). (*ii*) Vect. (C, H, U). (*iii*) Mult. (H, X, U, H). (*iv*) Mult. (H, U, X, H).

The "Regular Geometric Proposition," *R. Geom.* (C, H, X, U): (*i*) R. Vect. (C, H, X). (*ii*) R. Vect. (C, H, U). (*iii*) R. Mult. (H, X, U, H). (*iv*) R. Mult. (H, U, X, H). (*v*) R. Conj. (C, X, U). (*vi*) R. Conj. (C, U, X).

The "Normal Geometric Proposition," *N. Geom.* (C, H, X, U): (*i*) N. Vect. (C, H, X). (*ii*) N. Vect. (C, H, U). (*iii*) N. Mult. (H, X, U, H). (*iv*) N. Mult. (H, U, X, H). (*v*) N. Conj. (C, X, U). (*vi*) N. Conj. (C, U, X).

The "Gauge Proposition," *Gge.* (C, H, R, S): (*i*) For each r of R_G there is an s of S_G, such that $r.s = s.r = 1_C$, while for this s and any other element r' of R_G, $\|r'.s\| = \|s.r'\| \leq 1$. (*ii*) For each r_* of R_* there is an r of R_G, and a c of C_*, such that $r_* = c.r = r.c$.

The "Gauge Geometric Proposition," *Gge. Geom.* (C, H, X, U): (*i*) R. Geom. (C, H, X, U). (*ii*) N. Hyp. (C, H). (*iii*) There exists a subset X_G of X_*, and a subset U_G of U_*. (*iv*) Gge. (C, H, X, U). (*v*) Gge. (C, H, U, X).

The "Division Proposition," *Div.* (C, S, T): For each s of S_*, there is at least one t of T_*, such that (*i*) $s.t = t.s. = 1_C$, (*ii*) $\|s\|.\|t\| = 1$.

The "Norm Product Proposition," *N. Prod.* (R): For every r_1, r_2 of R, for which $\|r_1\|.\|r_2\|$ is defined, $\|r_1.r_2\| = \|r_1\|.\|r_2\|$.

A system (C, H, X, U) is said to be a *geometric system* if Geom. (C, H, X, U) is valid with respect to it. It is further said to *admit a gauge set*, if Gge. Geom. (C, H, X, U) also is valid. To *normalize* a geometric system is to define $\|x\|$ and $\|u\|$ in such a manner that N. Geom. (C, H, X, U) is valid, while any geometric system for which N. Geom. (C, H, X, U) holds is said to be a *normal geometric system.*

A geometric system admitting a gauge set may be normalized but leads to a special type of normal geometric system, since for any geometric system admitting a guage set, the following additional propositions may be proved: Div. (C, C, C), Div. (C, H, H), N. Prod. (C), N. Prod. (H), and for the normalized system obtained the following also are provable: Div. (C, X, U), Div. (C, U, X). For the general normal geometric system none of these six propositions may be true.

To normalize a geometric system admitting a guage set, define for every x_G and u_G, $||x_G|| = 1$, $||u_G|| = 1$, and for every $c.x_G$ and $c.u_G$, $||c.x_G|| = ||c||$, $||c.u_G|| = ||c||$. The theorem then follows without difficulty.

For the general normal geometric system, the *relative inclination, θ*, of two elements, x of X_*, and u of U_*, may be defined by, $||x.u|| = ||x||.||u|| \cos \theta$. Thus x of X_*, and u of U_* are mutually orthogonal if and only if $||x.u|| = O$. Two elements, x of X_* and u of U_*, are mutually *conjugate* if and only if $x.u = u.x = c.c$ while also $||x|| = ||u|| = ||c||$. Two elements, x of X_*, and u of U_*, are mutually *reciprocal* if and only if $x.u = u.x = 1_C$ while also $||x||.||u|| = 1$. The sets (x) and (u) for which $||x|| = 1$, and $||u|| = 1$, respectively, are Convex. The relation $||r_1 + r_2|| \leq ||r_1|| + ||r_2||$ is called the Triangle Inequality of the norm. The relation $||c.l|| = ||c||.||l||$ is called the Linearly Homogeneous Property of the norm. The expressions $||x||$ and $||u||$ are said to be Conjugate Norms and when continuous, each is a "Gauge Form." The expressions $x.u$ and $u.x$ are Inner Products. Each of the bilinear forms $x.u$ and $u.x$ "converges" if $||x||$ and $||u||$ are finite, since $||x.u|| \leq ||x||.||u||$, and $||u.x|| \leq ||u||.||x||$, which are statements of the general form of Schwarz's Inequality.

The following possibilities for a normal geometrical system may be emphasized: (*i*) The product among hypernumbers need not be commutative. (*ii*) The product involving two x's or two u's need not have a meaning. (*iii*) The system C may be an integral system in which division is not in general possible. (*iv*) $||n_C||$ may be less than n, where by n_C is meant $1_C + 1^C + \ldots + 1_C$ to n terms, for $n > 1$. (*v*) The conjugate of a given element need not be unique. (*vi*) The set of elements (x) for which $||x|| = 1$, may be dense but not continuous, as for example the rational points on a circle. In particular, the above theory is applicable to the Geometry of Numbers of Minkowski,[4] to a system where C is a Kürschak valuated field,[6] and to a system where H is a quaternion field.[12]

While in every normal geometric system each x of X_* has a conjugate, u of U_*, which has again the given x as its conjugate, the conjugate relation is not in general a simple one. The geometric systems for which (3′) is satisfied are identified by the semi-linear property that if x_1 and u_1 be conjugates, and x_2, u_2 be conjugates, then $c_1.x_1 + c_2x_2$ and $c'_1.u_1 + c'_2.u_2$ will be conjugates where c_1, c_2, c'_1, c'_2 are of C, while c and c' are themselves conjugates in a more elementary sense.

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[1] Due to Grassmann, *Geometrische Analyse*, 1847, paragraph 7, p. 16; cf. Gibbs, *Vector Analysis* (E. B. Wilson).

[2] Suggested by Gauss, Dirichlet, *Werke*, I, p. 539. Extended use by Kronecker, "De unitatibus complexis," *Werke* I, p. 14.

[3] Fréchet, M., Sur quelques points..., *Rend. Circ. M. Palermo*, **22,** 1906 (1–74).

Consult A. D. Pitcher and E. W. Chittenden, "On the Foundations of the Calcul Fonctionelle of Fréchet," *Trans. Amer. Math. Soc.*, **19,** 1918 (66–78).

[4] Minkowski, H., *Geometrie der Zahlen*, p. 9 (Ed. 1910), Leipzig

[5] Riesz, F., *Les systèmes d'equations linéaires*, Paris (1913).

[6] Kürschàk, J., Ueber Limesbildung. . ., *Crelle*, **142,** 1913 (211).

[7] Proved in elementary geometry for triangles, namely (1) above.

[8] Identified since C. Wessel (1799), Gauss, and Hamilton.

[9] Moore, E. H., *Fifth Int. Cong. Math.* **1912,** I (253).

[10] Moore, E. H., *Bull. Amer. Math. Soc.*, (Ser. 2), **18,** 1912 (334–362), and later papers.

[11] Cf. Moebius, Der barycentrische Calcul, (1897), *Werke*, I.

[12] Cf. Hurwitz, A., *Zahlentheorie der Quaternionen*, Berlin (1919).

PROBLEMS OF POTENTIAL THEORY

BY PROFESSOR G. C. EVANS

DEPARTMENT OF MATHEMATICS, RICE INSTITUTE

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1. *The Equations of Laplace and Poisson.*—As is well known, Bôcher considered the integral form of Laplace's equation:[1]

$$\int \frac{\partial u}{\partial n}\, ds = 0, \tag{1}$$

and showed that it was entirely equivalent to the differential form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{2}$$

for functions u continuous with their *first* partial derivatives over any "Weierstrassian" region (as Borel would call it[2]); in fact he showed that any such solution of (1) possessed continuous derivatives of all orders and satisfied (2) at every point. One can go still further, and consider solutions of (1) which have merely summable derivatives, and of the first order, with practically the same result.

THEOREM I.—If the function u is what we shall call a "potential function for its gradient vector ∇u,"[3] the components of the latter being summable superficially in the Lebesgue sense, and if the equation

$$\int \nabla_n u\, ds = 0 \tag{1}$$

is satisfied for every curve of a certain class,[4] then the function u has merely unnecessary discontinuities, and when these are removed by changing the value of u in the points at most of a point set of superficial measure zero, the resulting function has continuous derivatives of all orders and satisfies (2) at every point.

Let us pass on to the equation

$$\int \frac{\partial u}{\partial n}\, ds = \int \rho\, d\sigma \tag{3}$$

in which a curvilinear integral on the left is equal to a superficial integral on the right, and this equality is a generalization of Poisson's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \rho(x,y). \tag{4}$$

But we need not limit ourselves in the right hand member to quantities which are the integrals of functions ρ; instead we may take the right hand member to be any sort of additive function of point sets $f(e)$[5] (not necessarily absolutely continuous as in (3) or even continuous), or any sort of additive function of curves in the plane $f(s)$,[6] of limited variation. More precisely, we consider the equation

$$\int \nabla_n u\, ds = F(s) \tag{3'}$$

in which $F(s)$ is an additive function of curves in the plane, of limited variation, with discontinuities "of the first kind."

The difference of any two solutions of (3′) is a solution of (1′), and, therefore, part of the study of (3′) is the study of a particular solution. Such a particular solution is found to be the function

$$u(M_1) = \frac{1}{2\pi}\int_\Sigma \log\frac{1}{r}\, df(e) \tag{5}$$

defined in terms of a Lebesgue-Stieltjes integral. In this equation M_1 is the point of coördinates x_1, y_1, M the point of coördinates x, y which is the argument of the integration, and r the distance between them; in measuring angles the sense of r is taken as M_1M. This function may be shown to be the potential function of a gradient vector of which the component in the direction α is given by

$$\nabla_\alpha u(M_1) = \frac{1}{2\pi}\int_\Sigma \frac{\cos \alpha r}{r}\, df(e). \tag{6}$$

The vector given by (6) may be shown to be a solution of (3′) for every curve of the class Γ (specified below).

In two dimensions, a Stieltjes integral may be defined equally well in terms of both sorts of functional, and a function of curves used as well as a function of point sets for the function of limited variation. And thus if Σ' is a region enclosed in Σ, with boundary S' (a curve of class Γ), we may speak of the function

$$u'(M_1) = \frac{1}{2\pi}\int_{\Sigma'} \log\frac{1}{r}\, dF(s), \tag{5'}$$

which is identical with the function given by (5) for all points interior to Σ', provided that the equality

$$F(s) = f(e) \quad (e = \text{set of points inside } s), \tag{5''}$$

holds for any s of Γ on which $F(s)$ is continuous as a function of curves.

Methods of making correspondences of type (5″) will be discussed below.

2. *Various Forms of Green's Theorem.*—Let $F(s)$ and $G(s)$ be two functions of curves with discontinuities of the first kind, and $U(M)$ and $V(M)$ two solutions of the equations

$$\int \nabla_n U ds = F(s)$$

$$\int \nabla_n V ds = G(s),$$

which may be written in the form given by (5) or (5′) plus harmonic functions (i. e., solutions of (1′) after the unnecessary discontinuities have been removed). With respect to $F(s)$ and $G(s)$ let us introduce the symbols $T_F(s)$ and $T_G(s)$ to stand for their total variation functions. Then we have the following rather general result.

Theorem II.—If the discontinuities of $F(s)$ and $G(s)$ are of the first kind, a sufficient condition that the equation

$$\int_\sigma (\nabla_x U \nabla_x V + \nabla_y U \nabla_y V) d\sigma = \int_\sigma U dG(s) - \int_s U \nabla_n V ds$$
$$= \int_\sigma V dF(s) - \int_s V \nabla_n U ds \qquad (7)$$

shall hold for all regions σ and all boundaries s of Γ, internal to Σ', is the existence of the quantity

$$\int_{\Sigma'} \log \frac{1}{r'_{M_1M}} dT_F(s) dT_G(s).$$

A vector φ which satisfies the equation corresponding to (3′)

$$\int_s \varphi_n ds = F(s) \qquad (8)$$

for every curve of Γ may be called a *polarization vector* for the distribution $F(s)$; it is not necessary that such vectors shall, as in the examples already given, have potential functions. Those that we have already given satisfy also certain restrictions of integrability which we denote by condition N, as follows:

Condition N. Any component of φ is summable superficially, and the normal component is summable along any curve of Γ; in particular, the integral of the absolute value of φ_n, along any curve of Γ or finite number of mutually exclusive curves remains finite, less than some fixed number N, provided that the total length of the curves remains finite, less than some fixed number s_0.

We have then the following theorems:

Theorem III.—Let $G(s)$ be an arbitrary function of curves, additive and of limited variation,[7] and let $\psi(M)$ be a polarization vector for it which satisfies condition N; further let $u(M)$ be a function continuous over Σ' with its first partial derivatives. Then for every s of class Γ the following equation is valid:

$$\int_\sigma u dG(s) = \int_s u \psi_n ds + \int_\sigma \left(\frac{\partial u}{\partial x} \psi_x + \frac{\partial u}{\partial y} \psi_y \right) d\sigma \qquad (9)$$

Theorem IV.—If $\psi(M)$ satisfies condition N and $u(M)$ is continuous and a potential function for its vector gradient in Σ, the equation

$$\int_\sigma u dG(s) = \int_s u\psi_n ds + \int_\sigma (\nabla_x u\psi_x + \nabla_y u\psi_y) d\sigma \qquad (9')$$

remains valid provided that one of the following hypotheses (α), (β) or (γ) be imposed:

(α) $\nabla u(M)$ is bounded.

(β) $\psi(M)$ is bounded.

(γ) The quantities $\{\nabla u(M)\}^2$ and $\{\psi(M)\}^2$ are summable superficially.

If in this theorem we put $u(M) = \log 1/r$, we obtain:

THEOREM V.—If $\psi(M)$ satisfies condition N the equation

$$\int_\sigma \log \frac{1}{r} dG(s) = \int_s \log \frac{1}{r} \psi_n ds + \int_\sigma \left\{ \psi_x \frac{\partial}{\partial x} \log \frac{1}{r} + \psi_y \frac{\partial}{\partial y} \log \frac{1}{r} \right\} d\sigma \qquad (10)$$

is valid, given the region σ, for all points M_1 except possibly those which form a point set of superficial measure zero.

One or two more special theorems are perhaps interesting. If in Theorem II we put $V = \log 1/r$, we obtain the equation

$$2\pi U(M_1) = \int_s \left\{ U \frac{\partial}{\partial n} \log \frac{1}{r} - \log \frac{1}{r} \nabla_n u \right\} ds + \int_\sigma \log \frac{1}{r} dF(s) \quad (11)$$

which holds whenever $U(M_1)$ is the derivative of its own superficial integral, and, therefore, except for the points of a set of superficial measure zero.

If in Theorem IV we put

$$\psi = \lambda\varphi$$

where λ is a scalar point function, so that the vector φ satisfies the equation

$$\int_s \lambda\varphi_n ds = G(s)$$

then equation (9′) reduces to the following:

$$\int_\sigma \lambda\{\nabla_x u\varphi_x + \nabla_y u\varphi_y\} d\sigma = -\int_s \lambda u\varphi_n d\sigma + \int_\sigma u dG(s). \qquad (12)$$

These theorems provide for the Stieltjes integral with respect to a function of curves a representation in terms of a Lebesgue integral and a curvilinear integral. The curvilinear integral is essentially a curvilinear integral, depending only on the contour, and differs thus from the integral around general boundaries defined by P. J. Daniell,[9] which is a "frontier integral" not uniquely determined by the contour itself, but by some superficial point set.

The theorems of this section remain true if for σ is substituted a set of points measurable superficially in the sense of Borel, for s its frontier, and for $F(s)$, $G(s)$ the corresponding additive functions of point sets $f(e)$, $g(e)$, the frontier integrals being defined according to the method of Daniell. We shall come later to another possible method of determining frontier integrals.

3. *Definitions and Examples.*—It is desirable to render more precise some of the notions which we have used.

A curve of class Γ is a simple closed rectifiable curve composed of a finite number of pieces, each piece having a definite direction at every point; for each curve there is a constant Γ such that the following inequality holds.

$$\int_s \frac{|\cos nr|}{r_{M_1M}} ds < \Gamma,$$

wherever the point M_1 may be. For a curve which is everywhere convex, for instance, Γ may be taken as 2π.

With respect to gradient vector and potential function we have the following definition. If $A(M)$ is a vector point function whose components in two fixed directions (and therefore in any fixed direction) are summable, and $u(M)$ is a scalar point function whose integrals may be defined along any curve of class Γ, then A is spoken of as a *gradient* of u, and u as a *potential function* of A provided the equation

$$\int_\sigma A_\alpha d\sigma = \int_s u d\alpha' \tag{13}$$

is satisfied for every s of Γ and for every α, the subscript α being used to represent a fixed direction and α' that direction advanced by $\pi/2$.

It is sufficient that (13) should hold for two distinct directions α in order to hold for every direction.

The curvilinear integral in (13) is an example of what may be called an *absolutely continuous* function of curves. For the purpose of this article we say that an additive function of curves is of *limited variation* if it is the difference of two additive functions of curves, each one positive or zero for every curve of class Γ.

Every curve s of Γ determines a set e of points interior to the region σ which it bounds. We shall consider correspondences between functions of curves $F(s)$ and functions of points sets $f(e)$ such that for every curve on which $F(s)$ is continuous the equation

$$f(e) = F(s) \tag{14}$$

holds. Given $F(s)$ it may be shown that $f(e)$ is uniquely determined over all sets measurable in the sense of Borel, simply by means of (14).

On the other hand the equation (14) is not sufficient to determine $F(s)$, given $f(e)$. The discontinuities of $F(s)$ may still be distributed in various ways. The $F(s)$ which is defined by the formula

$$F(s_1) = \frac{1}{2\pi}\int_{s_1} ds_1 \int_\Sigma \frac{\cos nr}{r} df(e) \tag{15}$$

has all of its discontinuities of the *first kind;* that is, if a portion, s', which has everywhere a tangent, forms part both of s_1 and s_2, otherwise mutually exterior, a possible discontinuity upon s' would be shared equally by

$F(s_1)$ and $F(s_2)$; and if a point P is a vertex for s_1, a portion of the possible discontinuity at P proportional to the angle at the vertex is taken up by $F(s_1)$.

In order to illustrate this correspondence, take the following example. Let Σ be the square $0<x<1$, $0<y<1$; and to the points M^i of rational coördinates in this Σ assign positive numbers $p(M^i)$ in such a way that when the points are arranged in countable order, the series of numbers will be convergent; and make the definition

$$f(e) = p(e) = \Sigma_i p(M^i), \tag{16}$$

where the sum is extended over all the M^i in the set e. Introduce now an auxiliary function $\mu(\theta,M)$, which for a given point M depends on an angle θ made with a fixed direction, and which is such that the quantity $\int_0^{2\pi} \mu(\theta,M)d\theta$ is everywhere unity. Consider the quantity

$$H(M,\theta_1,\theta_2) = \int_{\theta_1}^{\theta_2} \mu(\theta, M)d\theta.$$

For the purpose of defining $F(s)$ every point of s may be considered temporarily as a vertex, the tangents forward and backward from M making, respectively, angles θ_1,θ_2 with the fixed direction. Then the equation

$$F(s) = \Sigma'_\sigma p(M^i) + \Sigma_s H(M^i,\theta_1,\theta_2)p(M^i),$$

where the first summation takes all the points M^i interior to σ and the second takes all the points M^i on the boundary s, satisfies the equation of correspondence (14) and defines an additive function of curves of limited variation which is discontinuous on every curve which contains a rational point. In general, the discontinuities of $F(s)$ will not be of the first kind; in order to make them so, and thus satisfy the equation (15), it is sufficient to put $\mu(\theta,M) = 1/2\pi$.

The directional derivative of a point function $u(M)$ is not necessarily a vector, merely on the basis of its existence. If, however, this derivative for two directions, say $\partial u/\partial x$ and $\partial u/\partial y$, is summable superficially and satisfies for these two directions and all curves of Γ the equation (13), then the vector which has these two quantities for its components in the two directions will be a gradient vector, and the original function its potential function. Yet it is simpler for the purpose of proving theorems and less artificial, to use directly a generalized partial derivative which has itself vectorial properties.

We say that $D_\alpha u$, *the generalized derivative in the direction* α of $u(M)$, is the limit, if such limit exists, of the expression

$$D_\alpha u = \lim_{\sigma=0} \frac{1}{\sigma} \int_s u d\alpha' \tag{17}$$

where the fixed direction α' makes an angle of $\pi/2$ with the fixed direction α, and σ denotes the area enclosed by s; it is understood that σ tends toward 0 only in such a way that the ratio σ/d^2, where d is the diameter of σ, remains different from zero by some positive quantity.

It is a theorem that the expression (6) is the generalized derivative of the expression (5) except at the points of a set E_0 of superficial measure zero, which may be chosen independently of the direction α. The function defined by (5) is therefore a solution of the equation

$$\int_s D_n u\, ds = F(s) \tag{18}$$

for any curve which does not contain points of E_0 of more than linear measure zero. The transition from this equation to the equation

$$\int_s \frac{\partial u}{\partial n}\, ds = F(s)$$

involving derivatives in the customary sense, is possible for classes of curves defined in terms of curvilinear coördinates, by means of the identity between a double and an iterated integral in these coördinates.

4. *A Double Integration by Parts.*—The Green's theorem from the point of view of section 2 may be regarded as an integration by parts of the multiple Stieltjes integral with respect to one dimension only. It is perhaps worth while to state the theorem which one obtains by a complete integration by parts of the double integral.

Let $u(M)$ be a continuous point function of limited variation[10] and let $q(M) = G(\bar{s})$ where $\bar{s}$ is the rectangle whose diagonally opposite vertices are M and the origin. Then if s is any simple closed regular curve, and points P_i with coördinates x_i, y_i are taken round the curve such that $|x_{i+1} - x_i| < \delta$, $|y_{i+1} - y_i| < \delta$ the following integrals have meaning:

$$\begin{aligned}\int u d_y q &= \lim_{\delta = 0} \Sigma_i u(x_i, \eta_i) \{q(x_i, y_{i+1}) - q(x_i, y_i)\} \\ &\qquad\qquad y_i \lesseqgtr \eta_i \lesseqgtr y_{i+1}, \\ \int u d_x q &= \lim_{\delta = 0} \Sigma_i u(\xi_i, y_i) \{q(x_{i+1}, y_i) - q(x_i, y_i)\} \\ &\qquad\qquad x_i \lesseqgtr \xi_i \lesseqgtr x_{i+1},\end{aligned} \tag{19}$$

and the identity

$$\int_\sigma u dq = \int_\sigma q du + \int_\sigma dqu - \int_s u d_y q + \int_s u d_x q \tag{19'}$$

is valid.[11] In this equation the quantity $\int dqu$ stands for the total variation of qu over the region σ.

5. *Applications.*—Consider the function

$$W(z_1) = \frac{-1}{2\pi i} \int_\Sigma \frac{df(e)}{z - z_1} \tag{20}$$

where z denotes the quantity $x + iy$, where $f(e) = f_1(e) + if_2(e)$ is a complex additive function of point sets, and the integral has the meaning:

$$W(z_1) = U(M_1) + iV(M_1)$$

with

$$U(M_1) = \frac{1}{2\pi}\int \frac{y-y_1}{r^2}\,df_1(e) - \frac{x-x_1}{r^2}\,df_2(e)$$

$$V(M_1) = \frac{1}{2\pi}\int \frac{x-x_1}{r^2}\,df_1(e) + \frac{y-y_1}{r^2}\,df_2(e).$$

From this evaluation follows the equation

$$\int_{s_1} W(z_1)dz_1 = F(s_1), \tag{21}$$

where $F(s)$ is the function of curves with discontinuities of the first kind which is equivalent to $f(e)$.

If we extend the notion of generalized derivatives to functions of a complex variable, it turns out that what corresponds to the quantity

$$\frac{\partial W}{\partial x} + i\frac{\partial W}{\partial y}$$

is the limit:

$$\frac{\lim}{\sigma=0}\ \frac{1}{i\sigma}\int W(z)dz,$$

which is merely the superficial derivative, where it exists, of the additive function of point sets $f(e)/i$. But this quantity exists in Σ except possibly at the points of a set of superficial measure zero. At a point where this superficial derivative itself vanishes it seems opportune to follow the direction of Borel's concept, and speak of the function $W(z)$ as *monogenic*.

If $\kappa(z)$ is a function of a complex variable, analytic in Σ, and $Q(z)$ is a solution of the equation

$$\int_s Q(z)dz = F(s)$$

and equal to $W(z)$ plus an analytic function, the equation

$$\int_s \kappa Q dz = \int_\sigma \kappa dF(s)$$

is satisfied. In particular, by taking as $\kappa(z)$ the function $1/(z-z_1)$, it may be deduced that except possibly at the points of a set of superficial measure zero the following identity holds:

$$Q(Z_1) = \frac{1}{2\pi i}\int_s \frac{Q(z)}{z-z_1}\,dz + \frac{1}{2\pi i}\int_\sigma \frac{dF(s)}{z-z_1} \tag{22}$$

The function given by (20) is the unique derivative of a certain functional of open curves in the plane, considered as a function of the end-point of the curve. We have, in fact, $W(z) = dZ(1|z)/dz$, where

$$Z(1|z_1) = \frac{1}{2\pi i}\int_\Sigma \log\,(z-z_1)df(e). \tag{23}$$

Here the values of arctan $y-y_1/x-x_1$ are taken, say, as the principal values for a particular value of $z_1 = z_0$, thus defining log $(z-z_0)$ for z in Σ; log $(z-z_1)$ is then defined in general by means of the curve 1 which connects z_0 with z_1.

A different sort of application of these ideas is to the Dirichlet problem: to determine a harmonic function throughout a Weierstrassian region by assigned frontier values. For this purpose consider an arbitrary simply connected T region as defined by Osgood[12] whose boundary consists of more than one point; for simplicity we may as well restrict it to the finite domain.

Let M_0 be an interior point of T and $g(M_0,M)$ its Green's function. Let $h(M_0,M)$ be the conjugate function of $g(M_0,M)$. These functions are both harmonic at every point of T, the point M_0 excepted; and the following facts can be stated with reference to the frontier of T:

To an accessible point on the frontier corresponds a value of h, $h = C$ (and perhaps more than one value), but to two different accessible frontier points cannot correspond the same value C.[13] We see then that if a value of h corresponds to an accessible frontier point, the point is uniquely determined. But it can be shown that all values of h except possibly some which constitute a set of measure zero actually do correspond to accessible points on the frontier.

For a given point M_0 of T we thus set up a correspondence between the curves $h\ (M_0,M) = C$ and the accessible points on the frontier of T.

Consider functions $u(M)$ which are limited and continuous in T, and whose first partial derivatives with their squares are summable over T. Form the integral

$$-\int_{P_1}^{P_2} u(M)\,\frac{\partial g}{\partial n}\,ds = \int_{h_1}^{h_2} u(M)dh(M_0,M) \qquad (24)$$

along a curve joining a point, P_1, on $h = C_1$ to a point, P_2, on $h = C_2$ and composed of points M given by the relation $g(M_0,M) = \psi(h)$, where $\psi(h)$ is an arbitrary continuous function of values of h between C_1 and C_2. As $\psi(h)$ is let approach zero remaining always not-negative and less than some finite N, it is seen by comparison with additive functions of point sets that the integral (24) has a limiting value; and the Stieltjes integral in the right hand member defines in the limit an absolutely continuous function $U(h)$ of h_2, which is therefore the integral of its derivative with respect to h, this derivative being a summable function $\bar{u}(h)$. It may be shown that as M travels along a curve, $h = C$, we have $\lim u(M) = \bar{u}(C)$ except for values of C which constitute a set of linear measure zero. In other words, the frontier values of $u(M)$, taken in this sense, are summable with respect to h, and the frontier integral is properly a Stieltjes integral of frontier values.

Vice versa, given $\bar{u}$ limited and summable with respect to h on the frontier of T there is one and only one bounded function $u(M)$, continuous and harmonic in T, with first partial derivatives and their squares summable over T, and such that it takes on the frontier values $\bar{u}(h)$ "almost everywhere" in the above-described manner.

No single point M_0 need be preferred. If $\bar{u}(M)$ is summable with respect to $h(M_0, M)$ on the frontier of T when M_0 is some definite point in T, it is summable on the frontier of T with respect to $h(M_1, M)$, M_1 being any point of T.[14]

[1] *Proc. Amer. Acad. Sci., Boston,* **41,** 1905–06.

[2] Borel, E., *Leçons sur les fonctions monogènes,* Paris, 1917, p. 10.

[3] Evans, *Rend. R. Accad. Lincei,* (Ser. 5), **28,** 1919 (262–265).

[4] For example it is sufficient to consider merely rectangles. See also the class Γ below.

[5] de la Vallée Poussin, C., *Integrales de Lebesgue,* Paris, 1916, p. 57.

[6] Volterra, V., *Leçons sur les fonctions de lignes,* Paris, 1913, chapt. II. It is to be noted that the definitions of continuity in the case of functions of curves and in the case of functions of point sets do not correspond at all to the same situation.

[7] The discontinuities need not be of the first kind.

[8] By $\{\psi(M)\}^2$ is meant the quantity $\{\psi_x(M)\}^2 + \{\psi_y(M)\}^2$.

[9] *Bull. Amer. Meth. Soc.,* **25,** November, 1918 (65–68).

[10] de la Vallée Poussin, C., *loc. cit.,* p. 98; *Trans. Amer. Math. Soc.,* **16,** 1915 (493).

[11] This identity is given for the special case of a rectangle, where the integrals (19) do not have to be used, by W. H. Young, *Proc. London Math. Soc.,* (Ser. 2), **16,** 1917.

[12] Osgood, *Lehrbuch der Funktionentheorie,* Leipzig, 1912, p. 151.

[13] Osgood and Taylor, *Transactions of the American Mathematical Society,* **14,** 1913 (277–298).

[14] A complete exposition of the theorems of this paper will appear in an early issue of the *Rice Institute Pamphlet.*

THERMO-ELECTRIC ACTION AND THERMAL CONDUCTION IN METALS: A SUMMARY

By Edwin H. Hall

Jefferson Physical Laboratory, Harvard University

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I shall now undertake to show in brief what I have accomplished in the series of papers which for some years I have been publishing in these Proceedings in relation to thermo-electric action and thermal conduction in metals.

Starting with the hypothesis of dual electric conduction, that is, conduction maintained in part by the passage of electrons from atomic union to atomic union during the contacts of atoms with positive ions, and in part by "free" electrons in the comparatively weak fields of force called the inter-atomic spaces, I derived[1] an equation giving the conditions of steady state in a detached bar hot at one end and cold at the other, and two equations for the conditions of equilibrium at a junction of two metals.

For the detached bar the steady state is a current of associated electrons up the temperature gradient and an equal current of free electrons down the temperature gradient. This involves a freeing of electrons from atomic unions, ionization, at the hot end of the bar, with absorption of heat, and re-association at the cold end, with evolution of heat. Evidently this is a process of conveying heat, and the question at once arises whether it is capable of accounting for the whole or any considerable part of thermal conduction in metals.

At the junction of two metals, on the other hand, the conditions of equilibrium permit no cyclic or circulatory motion of the electrons.

In a later paper[2] I introduced the following assumptions: That n, the number of free electrons per cu. cm. of the metal, is expressed by the equation

$$n = zT^q, \tag{1}$$

where T is absolute temperature while z and q are constants. That $(k_f \div k)$, the ratio of free-electron conductivity to total conductivity, is given by the equation

$$(k_f \div k) = C + C_1t + C_2t^2, \tag{2}$$

where t is temperature on the ordinary centigrade scale and C, C_1, C_2, are constants.

That the heat of ionization, in ergs, per electron is

$$\lambda' = \lambda_c' + 2.5RT, \tag{3}$$

where λ_c' is a constant[3] and R is the gas-constant for a single molecule.

In a still later paper[4] I rejected (3) and put in its place

$$\lambda' = \lambda_c' + sRT, \tag{3'}$$

where s is a constant never less than 2.5. This change involved the revision of all the tables of numerical values contained in the paper in which equation (3) was used.

Equations (1), (2) and (3′) introduced for each metal seven characteristic constants, z, q, C, C_1, C_2, λ_c' and s, but z may for the most part be left out of consideration, as the definite numerical value of n is needed only incidentally[5] in what follows.

To determine the six other constants for any metal we need six equations. Let us suppose that we know for each metal the value of the Thomson effect, σ, the value of the heat conductivity at 0° C., θ_0, and at 100° C., θ_{100}.

My expression for σ is in the form

$$\sigma = K + (K_1 + K_2t)T, \tag{4}$$

in which K, K_1 and K_2 are determined functions of q, C, C_1, etc. Bridgman's experiments give me the values of K, K_1 and K_2, for many metals, and so from the Thomson effect I get three equations, which are too long to be given here.

My expression for thermal conductivity is

$$\theta = \frac{R^2}{e^2}\left(\frac{\lambda_c'}{R} + sT\right)\frac{k_a k_f}{k}(s + q - 1.5), \tag{5}$$

in which k_a, the "associated-electron" conductivity, is $(k - k_f)$. With the known values of θ_0 and θ_{100} I get two more equations involving the six characteristic quantities to be determined for each metal.

Thus for two metals, α and β, with twelve characteristic quantities to be determined, I have ten equations given by the properties of the metals, each metal being considered separately. When the two metals are brought into contact at 0° C. and again at 100° C., I have two Peltier-effect equations of the form

$$\Pi = \left(\frac{k_f}{k}\right)_\beta \lambda_\beta - \left(\frac{k_f}{k}\right)_\alpha \lambda_\alpha \qquad (6)$$

one for each temperature, where λ_β is the heat, in ergs, required to free $(1 \div e)$ electrons within β, and λ_α has a like meaning for α.

Thus I have twelve equations to determine the twelve constants in question.

If my theory were perfect and my experimental data complete, every constant could be found exactly, or to any desired degree of precision, by means of the equations here mentioned; but neither of these conditions is fulfilled. It is unlikely, for example, that equation (1) holds strictly through any considerable range of temperature. The experimental data I have used are doubtless incorrect to some extent, and in several places I have had to fill observational gaps by rather hazardous methods, particularly in values of thermal conductivity. Somewhat arbitrarily I have chosen to treat Bridgman's values of σ as strictly correct, though he does not so regard them. I have done the same with regard to the available values of k_o, k_{100}, θ_o, and Π_o. Accordingly the values of θ_{100} and Π_{100} worked out from my equations are subject to the accumulated errors of theory and data, and it is not very disconcerting to find them differing considerably in some cases from the observed values or those indicated by other evidence.

If the question is raised why values of the constants cannot be found that will satisfy all of the equations, even if some of the data are inaccurate, the answer is that not all values are available for these constants, thus C, q, λ_c', and s must all be positive quantities.

Below are given for each of many metals, and for two alloys, the values of the six constants that I have hit on to account, as well as may be, for the observed or estimated values of the Thomson effect, thermal conductivity, and Peltier effect with respect to bismuth, in these substances. It is to be remembered that the algebraic sign which I give to the Thomson effect in any metal is opposite to the one ordinarily used; for example, I call σ positive in iron and negative in copper. The Peltier effect, Π, is given as the amount of heat, in ergs, absorbed by $(1 \div e)$ electrons in going from the metal in question to bismuth. Thermal conductivities are given in calories/cm., sec., deg. C. The constant part, λ_c', of λ' is given in ergs; δ_c is the "ionizing potential," in volts, corresponding to λ_c'; δ_o and δ_{100}, respectively, are the ionizing potentials, in volts, for the total work of ionization at 0° C. and at 100° C.; k is electric conductivity in absolute measure. The numbers in the horizontal lines marked *Calc.*

are those which my formulae yield when the given values of the constants are put into them.

ALUMINIUM

C	$C_1 \times 10^6$	$C_2 \times 10^3$	q	s	$\lambda'_C \times 10^{16}$	δ_C	δ_o	δ_{100}
0.0846	−2.85	6.19	1.45	5.64	176	0.011	0.142	0.190
	σ (in ergs/° C.)		θ_0	θ^{I}_{00}	$\Pi_o \times 10^{-4}$	$\Pi_{100} \times 10^{-4}$	$k \times 10^6$	$k_{100} \times 10^6$
Data	$(-0.016 + 0.006t)T$		0.477	0.492	202	263	380	265
Calc.	$(-0.016 + 0.006t)T$		0.478	0.445	201	210		

BISMUTH

C	$C_1 \times 10^6$	$C_2 \times 10^8$	q	s	$\lambda'_C \times 10^{16}$	δ_C	δ_o	δ_{100}
0.19^0	−80.4	0	1.2	4.35	1088	0.068	0.169	0.206
	σ (ergs/° C.)		θ_o	θ_{100}	$\Pi_o \times 10^{-4}$	$\Pi_{100} \times 10^{-4}$	$k_o \times 10^6$	$k_{100} \times 10^6$
Data	$-3.2T$		0.0202	0.0161	0	0	9.30	6.46
Calc.	$-3.2T$		0.0201	0.0164	0	0		

COBALT

C	$C_1 \times 10^6$	C_2	q	s	$\lambda'_C \times 10^{16}$	δ_C	δ_o	δ_{100}
0.134	175	0	1.55	5.26	32	0.002	0.125	0.170
	σ (ergs/° C.)		θ_o (6)	θ_{100} (6)	$\Pi_o \times 10^{-4}$	$\Pi_{100} \times 10^{-4}$	$k_o \times 10^6$	$k_{100} \times 10^6$
Data	$7.8T$		0.161(?)	0.164(?)	156	172	102	74.7
Calc.	$7.8T$		0.160	0.177	154	117		

COPPER

C	$C_1 \times 10^6$	C_2	q	s	$\lambda'_C \times 10^{16}$	δ_C	δ_o	δ_{100}
0.068	−16.6	0	1.49	6.8	64	0.004	0.162	0.221
	σ (ergs/° C.)		θ_o	θ_{100}	$\Pi_o \times 10^{-4}$	$\Pi_{100} \times 10^{-4}$	$k_o \times 10^6$	$k_{100} \times 10^6$
Data	$-0.966T$		0.920	0.908	211	280	650	455
Calc.	$0.966T$		0.928	0.866	211	229		

GOLD

C	$C_1 \times 10^6$	$C_2 \times 10^9$	q	s	$\lambda'_C \times 10^{16}$	δ_C	δ_o	δ_{100}
0.070	−16.3	8.7	1.49	6.7	48	0.003	0.159	0.217
	σ (ergs/° C)		θ_o	θ_{100}	$\Pi_o \times 10^{-4}$	$\Pi_{100} \times 10^{-4}$	$k_o \times 10^6$	$k_{100} \times 10^6$
Data	$(-0.934 + 0.00t)T$		0.699	0.703	211	280	490	351
Calc.	$(-0.934 + 0.00t)T$		0.696	0.666	210	227		

IRON

C	$C_1 \times 10^6$	$C_2 \times 10^3$	q	s	$\lambda' c \times 10^{16}$	δ_C	δ_0	δ_{100}
0.02	12.1	19.9	1.2	15	128	0.008	0.348	0.487
	σ (ergs/° C.)		θ_0	θ_{100}	$\Pi_0 \times 10^{-4}$	$\Pi_{100} \times 10^{-4}$	$k_0 \times 10^6$	$k_{100} \times 10^6$
Data	$(1.78 + 0.0516t)T$		0.163	0.151	247	310	78	48.1
Calc.	$(1.78 + 0.0516t)T$		0.164	0.146	251	226		

MAGNESIUM

C	$C_1 \times 10^8$	$C_2 \times 10^8$	q	s	$\lambda' c \times 10^{16}$	δ_C	δ_0	δ_{100}
0.061	−13.8	0	1.499	6.8	608	0.038	0.196	0.255
	σ (ergs/° C.)		θ_0	θ_{100}	$\Pi_0 \times 10^{-4}$	$\Pi_{100} \times 10^{-4}$	$k_0 \times 10^6$	$k_{100} \times 10^6$
Data	$-0.008T$		0.376(?) (7)	?	203	266	240	?
Calc.	$-0.008T$		0.374		202	220		

MOLYBDENUM

C	$C_1 \times 10^6$	$C_2 \times 10^8$	q	s	$\lambda' c \times 10^{16}$	δ_C	δ_0	δ_{100}
0.053	−64.4	10.8	1.3	8	176	0.011	0.197	0.266
	σ (ergs/° C.)		θ_0	θ_{100}	$\Pi_0 \times 10^{-4}$	$\Pi_{100} \times 10^{-4}$	$k_0 \times 10^6$	$k_{100} \times 10^6$
Data	$(-4.33 + 0.015t)T$		0.43(8)	0.42(8)	219	319	274	191
Calc.	$(-4.33 + 0.015t)T$		0.43	0.358	217	252		

NICKEL

C	$C_1 \times 10^6$	C_2	q	s	$\lambda' c \times 10^{16}$	δ_C	δ_0	δ_{100}
0.132	84.9	0	1.6	5.0	176	0.011	0.127	0.170
	σ (ergs/° C.)		θ_0	θ_{100}	$\Pi_0 \times 10^{-4}$	$\Pi_{100} \times 10^{-4}$	$k_0 \times 10^6$	$k_{100} \times 10^6$
Data	$0.0356T$		0.143	0.138	155	187	93	62.5
Calc.	$0.0356T$		0.142	0.134	153	136		

PALLADIUM

C	$C_1 \times 10^6$	C_2	q	s	$\lambda' c \times 10^{16}$	δ_C	δ_0	δ_{100}
0.083	64.3	0	1.6	6.51	139	0.0087	0.160	0.216
	σ (ergs/° C.)		θ_0	θ_{100}	$\Pi_0 \times 10^{-4}$	$\Pi_{100} \times 10^{-4}$	$k_0 \times 10^6$	$k_{100} \times 10^6$
Data	$3.52T$		0.165	0.182	188	232	100	75.9
Calc.	$3.52T$		0.165	0.179	188	182		

PLATINUM

C	$C_1 \times 10^6$	C_2	q	s	$\lambda' c \times 10^{16}$	δ_C	δ_0	δ_{100}
0.0808	50.8	0	1.6	6.24	176	0.011	0.157	0.210
	σ (ergs/° C.)		θ_0	θ_{100}	$\Pi_0 \times 10^{-4}$	$\Pi_{100} \times 10^{-4}$	$k_0 \times 10^6$	$k_{100} \times 10^6$
Data	$2.67T$		0.164	0.173	195	245	110	79.3
Calc.	$2.67T$		0.167	0.170	194	195		

SILVER

C	$C_1 \times 10^6$	C_2	q	s	$\lambda' c \times 10^{16}$	δ_C	δ_0	δ_{100}
0.0663	−14.2	0	1.49	7.1	64	0.004	0.169	0.230
	σ (ergs/° C.)		θ_0	θ_{100}	$\Pi_0 \times 10^{-4}$	$\Pi_{100} \times 10^{-4}$	$k_0 \times 10^6$	$k_{100} \times 10^6$
Data	$-0.864T$		1.009	0.992	210	279	670	476
Calc.	$-0.864T$		1.018	0.964	209	226		

THALLIUM

C	$C_1 \times 10^6$	$C_2 \times 10^8$	q	s	$\lambda' c \times 10^{16}$	δ_C	δ_0	δ_{100}
0.0625	3.48	2.69	1.48	7.3	160	0.01	0.180	0.243
	σ (ergs/° C.)		θ_0	θ_{100}	$\Pi_0 \times 10^{-4}$	$\Pi_{100} \times 10^{-4}$	$k_0 \times 10^6$	$k_{100} \times 10^6$
Data	$(0.268 + 0.0034t)T$		0.090[8]	0.096[8]	208	271	57	43.9
Calc.	$(0.268 + 0.0034t)T$		0.090	0.094	209	221		

TIN

C	$C_1 \times 10^6$	C_2	q	s	$\lambda' c \times 10^{16}$	δ_C	δ_0	δ_{100}
0.0654	2.36	0	1.51	6.66	368	0.023	0.179	0.236
	σ (ergs/° C.)		θ_0	θ_{100}	$\Pi_0 \times 10^{-4}$	$\Pi_{100} \times 10^{-4}$	$k_0 \times 10^6$	$k_{100} \times 10^6$
Data	$0.134T$		0.157	0.145	204	266	105	72.6
Calc.	$0.134T$		0.154	0.142	204	220		

TUNGSTEN

C	$C_1 \times 10^6$	C_2	q	s	$\lambda' c \times 10^{16}$	δ_C	δ_0	δ_{100}
0.042	−35	0	1.48	11.4	40	0.0025	0.268	0.366
	σ (ergs/° C.)		θ_0 (9)	θ_{100} (9)	$\Pi_0 \times 10^{-4}$	$\Pi_{100} \times 10^{-4}$	$k_0 \times 10^6$	$k_{100} \times 10^6$
Data	$-341T$		?	?	207	285	140	106
Calc.	$-341T$		0.352	0.335	208	234		

ZINC

C	$C_1\times10^6$	C_2	q	s	$\lambda'_c\times10^{16}$	δ_c	$\delta_\circ$	δ_{100}
0.0613	16.7	0	1.55	7	240	0.015	0.178	0.238
	σ (ergs/° C.)		$\theta_\circ$	θ_{100}	$\Pi_\circ\times10^{-4}$	$\Pi_\circ\times10^{-}$	$k_\circ\times10^4$	$k_{100}\times10^{16}$
Data	$0.99T$		0.266	0.262	211	274	180	127
Calc.	$0.99T$		0.266	0.258	212	225		

CONSTANTAN

C	$C_1\times10^6$	C_2	q	s	$\lambda'_\circ\times10^{16}$	δ_c	$\delta_\circ$	δ_{100}
0.135	141	0	1.55	6.65	48	0.003	0.158	0.215
	σ (ergs/° C.)		$\theta_\circ$	θ_{100}	$\Pi_\circ\times10^{-4}$	$\Pi_{100}\times10^{-4}$	$k_\circ\times10^6$	$k_{100}\times10^6$
Data	$7.94T$		0.052	0.064	108	106	20.4	20.4
Calc.	$7.94T$		0.052	0.076	108	55		

MANGANIN

C	$C_1\times10^6$	$C_2\times10^8$	q	s	$\lambda'_\circ\times10^{16}$	δ_c	$\delta_\circ$	δ_{100}
0.050	−98	4.07	1.51	9.65	54	0.0034	0.228	0.311
	σ (ergs/° C.)		$\theta_\circ$	θ_{100}	$\Pi_\circ\times10^4$	$\Pi_{100}\times10^4$	$k_\circ\times10^6$	$k_{100}\times10^6$
Data	$(-0.828-0.0067t)T$		0.051	0.063	207	275	23.8	23.8
Calc.	$(-0.828-0.0067t)T$		0.051	0.068	206	223		

The preceding tabulation deals with every metal and alloy for which I have the necessary data, except cadmium, and for this metal, as I have shown elsewhere,[10] there is some reason for doubting the correctness of the data, Bridgman's value for σ in cadmium being about ten times as large as that found by other investigators.

In the table that follows, γ is the estimated ratio, expressed in per cents, of the number of free electrons to the number of atoms, including the positive ions, within the metal at 0° C. In arriving at the given value of γ I have made use of the equation

$$k = \frac{nlce^2}{4RT} \tag{7}$$

for the free-electron conductivity. The second member differs from Drude's familiar expression for the total conductivity, supposed by him due entirely to the free electrons, only in having 4 instead of 6 in the denominator. I have taken l, somewhat arbitrarily but not without regard to what others have done in this particular, as equal to ten times the centre to centre distance of adjacent atoms.

In columns II and III are given, in absolute units, the values of the potential gradient urging the associated electrons up the temperature gradient, at 0° C. and 100° C., respectively. P is potential due to electric charge, and (dP/dT) seems open to electrostatic measurement. P_a is potential due to differential attraction of the unequally heated metal for the associated electrons, and (dP_a/dT) does not seem within reach of direct experiment at present. Integrated from 0° C. to 100° C., $d(P + P_a)/dT$ gives differences of potential ranging from about 0.0026 volt in iron to about 0.0065 volt in bismuth and in cobalt and about 0.008 volt in constantan.

Column IV gives values of the "ionizing potential," in volts, within the metals at 0° C., and column V shows what these values would be at 2727° C., or 3000° absolute, if the equation

$$\lambda' = \gamma'_c + sRT$$

held all the way to that temperature.

Nothing exact can be expected from such an extrapolation, which takes no account of melting or of vaporization, but it is interesting to note that the values of δ thus obtained are of the same order of magnitude as those found by direct experiment on the vapors of metals, though still somewhat smaller than these latter values.

Column VI shows, in calories, the amount of heat absorbed by the free ions and by the process of ionization in heating 1 cu. cm. of the metal 1° C. in the neighborhood of 0° C. The former of these quantities is $1.5R\gamma n$, and the latter $dn/dT.\lambda' = qn\lambda'/T$, each reduced to calories. Column VII gives the thermal capacity, per cu. cm., of the metals as found by experiment.

	I γ %	II $\left[\frac{d(P+Pa)}{dT}\right]_0$	III $\left[\frac{d(P+Pa)}{dT}\right]_{100}$	IV δ at 0° C.	V δ at 2727° C.	VI	VII
Aluminium	10.5	−4070	−4050	0.142	1.46	0.212	0.54
Bismuth	1.0	−6620	−6340	0.169	1.18	0.009	0.29
Cobalt	4.9	−6120	−6920	0.125	1.35	0.139	0.86
Copper	11.7	−3970	−3870	0.162	1.75	0.388	0.80
Gold	11.4	−4030	−3940	0.159	1.72	0.260	0.60
Iron	0.7	−2530	−2680	0.348	3.85	0.038	0.85
Magnesium	11.1	−3550	−3550	0.196	1.78	0.224	0.40
Molybdenum	4.4	−3540	−3110	0.197	2.06	0.099	0.70
Nickel	3.1	−5790	−6160	0.127	1.29	0.093	0.89
Palladium	2.6	−4720	−5080	0.160	1.68	0.069	0.57
Platinum	2.9	−4410	−4680	0.157	1.69	0.078	0.67
Silver	15.2	−4050	−3960	0.169	1.82	0.365	0.58
Thallium	1.8	−3910	−3960	0.180	1.88	0.027	0.36
Tin	3.2	−3700	−3720	0.179	1.73	0.052	0.39
Tungsten	1.9	−4120	−3770	0.268	2.92	0.072	0.64
Zinc	3.2	−3720	−3820	0.178	1.81	0.093	0.64
Constantan	0.7	−7780	−8590	0.158	1.71	0.024	0.85
Manganin	0.3	−4180	−4060	0.228	2.47	0.013	0.77

Evidently any number given in column VI should be smaller than the corresponding number given in column VII; and this is the case; but, for aluminium, copper, gold, and magnesium the ratio approaches one-half, and for silver it is greater than one-half. This is a larger value of the ratio than seems probable; that is, one might expect for these metals, if the values of column VI are true, greater departures from the law of Dulong and Petit than they really show. It is to be remembered, however, that the values of γ, upon which the numbers given in column VI depend, are based on the assumption that the mean free path of the free electrons within a metal is in all cases ten times the centre to centre distance of adjacent atoms. If we assume longer paths, we get correspondingly smaller values in columns I and VI. According to Compton and Ross[11] a photo-electrically excited electron can move in gold about twenty times the centre to centre distance of the atoms.

In view of the fact that the known specific heats of metals are in some cases smaller and in some cases larger than the law of Dulong and Petit requires, one may allow himself considerable latitude in dealing with the distribution of heat energy among the various degrees of freedom of the atoms. It is perhaps worth while to note that the numbers given in column VI would be larger if calculated for a temperature above 0° C., as this fact suggests at least an explanation of the general increase in the specific heats of metals with rise of temperature. More exact knowledge than we now have concerning the rate of this increase is greatly to be desired.

Doubtless my theory of heat conduction will be criticized as failing to deal in any general way with the Wiedemann-Franz law. This law is by no means exact. According to Jäger and Diesselhorst the ratio ($\theta \div k$) varies, at 18°C., from 6.36×10^{10} in aluminium to 9.64×10^{10} in bismuth, while the temperature coefficient of this ratio varies from 0.15% in bismuth to 0.46% in platinum and palladium. In dealing theoretically with the relations of thermal to electrical conductivity we may proceed in either of two ways. We may seek for such a mechanism of conduction and such a general formula as will give the W.-F. law and then undertake to show why the individual metals depart from it, or we may proceed at first without any regard to this law but finally treat each metal in such a way as to explain its more or less approximate conformity with the law. The latter is the method I have followed.

I do not claim complete success in this undertaking. Some of the expedients I have adopted—for example, the values of s which I have assumed—are open to doubt. It may be that the mechanism which I have imagined and described, convection of heat by means of a circulatory electric current, can account for only a part, perhaps a small part, of the total heat conduction of metals; but I trust that enough has been done to show that the possibilities of this mechanism are worthy of serious consideration.

I propose next to consider how far the ideas set forth in this paper will go toward explaining the various "transverse" electro-magnetic and hermomagnetic effects in metals.

[1] These Proceedings for April, 1918, pp. 98–103. I there used in turn "hypothesis A" and "hypothesis B" concerning the behavior of the free electrons as a gas. In nearly all that follows I have used "hypothesis A" which is that, aside from electric influences, the free electrons tend to equality of gas pressure throughout the bar, whereas "hypothesis B" assumes that they tend to follow the law of "thermal effusion."

[2] These Proceedings for March, 1920, pp. 139–154.

[3] Strictly, λ'_0 was used here instead of λ'_c, with the same meaning.

[4] These Proceedings, **6**, 1920 (617).

[5] It seems probable that in dealing with the conditions at the junction of two metals we really need to use the conception of number-density of free electrons in the "co-volume" of each metal. In fact, n may without harm be interpreted as having this meaning in all my formulas except the one, taken from Drude, in which I express the free-electron conductivity in terms of mean free path, mean velocity, etc., of the free electrons. See the latter part of this paper, where γ, the grade of ionization, is estimated.

[6] Estimated by use of the Wiedermann-Franz ratio (average for 12 metals) applied to k_o and k_{100}.

[7] Mean value, according to Lorenz (1881) between 0° C. and 100° C.

[8] See note under Cobalt.

[9] θ_{18} is given as 0.35 on the authority of Coolidge.

[10] These Proceedings, **6**, 1920 (617).

[11] *Physical Review*, May, 1919.

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QUATERNIONS AND THEIR GENERALIZATIONS

By Leonard Eugene Dickson

Department of Mathematics, University of Chicago

Read before the Academy, April 26, 1921

1. The discovery of quaternions by W. R. Hamilton in 1843 has led to an extensive theory of linear algebras (or closed systems of hypercomplex numbers) in which the quaternion algebra plays an important rôle. Frobenius[1] proved that the only real linear associative algebras in which a product is zero only when one factor is zero are the real number system, the ordinary complex number system, and the algebra of real quaternions. A much simpler proof has been given by the writer.[2] Later, the writer[3] was led to quaternions very naturally by means of the four-parameter continuous group which leaves unaltered each line of a set of rulings on the quadric surface $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0$.

The object of the present note is to derive the algebra of quaternions and its direct generalizations without assuming the associative or commutative law. I shall obtain this interesting result by two distinct methods.

2. The term *field* will be employed here to designate any set of ordinary complex numbers which is closed under addition, subtraction, multiplication, and division. Thus all complex numbers form a field, likewise all real numbers, or all rational numbers.

Just as a couple (a, b) of real numbers defines an ordinary complex number $a + bi$, where $i^2 = -1$, so also an n-tuple $(x_1, .., x_n)$ of numbers of a field F defines a hypercomplex number

$$x = x_1e_1 + x_2e_2 + \ldots . + x_ne_n, \tag{1}$$

where the *units* $e_1, \ldots, e_n$ are linearly independent with respect to the field F and possess a multiplaction table

$$e_ie_j = \sum_{k=1}^{n} \gamma_{ijk} e_k \qquad (i, j = 1, \ldots, n), \tag{2}$$

n which the γ's are numbers of F. Let $x' = \Sigma x'_i e_i$ be another hypercomplex number whose coördinates x'_i are numbers of F. Then shall

$$xx' = \sum_{i,j=1}^{n} x_i x'_j . e_i e_j,\ x \pm x' = \sum_{i=1}^{n} (x_i \pm x'_i) e_i,\ fx = xf = \sum_{i=1}^{n} fx_i . e_i,$$

when f is in F, so that multiplication is distributive. Under these assumptions, the set of all numbers (1) with coördinates in F shall be called a *linear algebra over F*.

3. We assume that e_1 is a principal unit (modulus), so that $e_1 x = x e_1 = x$ for every number x of the algebra, and write 1 for e_1. We assume that every number of the algebra satisfies a quadratic equation with coefficients in F. If $e^2 + 2ae + b = 0$, $(e + a)^2 = a^2 - b$, so that we may take the units to be 1, $E_2, \ldots, E_n$, where $E_i^2 = s_{ii}$, a number of F. For i and j distinct and >1, $E_i \pm E_j$ satisfies a quadratic, so that $(E_i \pm E_j)^2 = s_{ii} + s_{jj} \pm (E_i E_j + E_j E_i)$ is a linear function of $E_i \pm E_j$. Thus $E_i E_j + E_j E_i$ is a linear function of $E_i + E_j$ and of $E_i - E_j$, and hence is a number $2s_{ij} = 2s_{ji}$ of F.

Let $u_2, \ldots, u_n$ be arbitrary numbers of F and write $U = \Sigma u_k E_k$. Then U^2 equals

$$Q = \sum_{k_1,l=2}^{n} s_{kl} u_k n_l.$$

It is a standard theorem that Q can be reduced to $\Sigma c_i v_i^2$ by a linear transformation $u_k = \Sigma a_{kl} v_l$ with coefficients in F and of determinant $\neq 0$. Write

$$e_l = \sum_{k,l=2}^{n} a_{kl} E_k \qquad (l = 2, \ldots, n).$$

Then 1, $e_2, \ldots, e_n$ are linearly independent and may be taken as the new units of our algebra over F. Then

$$U = \sum_{k,l} a_{kl} v_l E_k = \sum_{l} v_l e_l,$$

$$U^2 = \sum_{k,l} v_k v_l e_k e_l = \sum c_i v_i^2.$$

Hence

$$e_i^2 = c_i,\ e_i e_j + e_j e_i = 0 \qquad (i, j = 2, \ldots, n,\ i \neq j). \qquad (3)$$

4. Write $x = x_1 + \Sigma x_i e_i$. Then $(x - x_1)^2 = \Sigma c_i x_i^2$. This gives $xx' = x'x = \sigma$, where

$$x' = 2x_1 - x = x_1 - \sum_{i=2}^{n} x_i e_i, \quad \sigma = x_1^2 - \sum_{i=2}^{n} c_i x_i^2.$$

We shall call x' the *conjugate* to x and $\sigma = \sigma(x)$ the *norm* of x. Hence the product of any number and its conjugate in either order equals its norm. We assume that the norm of a product equals the product of the norms of the factors:

$$\sigma(x)\sigma(\xi) = \sigma(X), \text{ if } x\xi = X, \tag{4}$$

and shall investigate the resulting types of linear algebras. We assume also that each $c_i \neq 0$ in (3).

5. By (2) the coördinates of $X = x\xi$ are $X_k = \Sigma x_i \xi_j \gamma_{ijk}$. Since $e_i^2 = c_i$, we have $\gamma_{ii1} = c_i$, $\gamma_{iik} = 0$ $(i>1, k>1)$. Hence

$$X_1 = x_1\xi_1 + \sum_{i=2}^{n} x_i\xi_i c_i + \sum_{\substack{i,j=2 \\ i \neq j}}^{n} x_i \xi_j \gamma_{ij1},$$

$$X_k = x_1\xi_k + x_k\xi_1 + \sum_{\substack{i,j=2 \\ i \neq j}}^{n} x_i\xi_j\gamma_{ijk}. \quad (k>1) \tag{5}$$

Since this transformation is the identity $X = x$ if $\xi = 1$, we obtain an infinitesimal transformation by taking $\xi_1 = 1$, $\xi_j = \delta t$, $\xi_i = 0 (i \neq 1, j)$:

$$\delta x_1 = X_1 - x_1 = \left\{ c_j x_j + \sum_{\substack{i=2 \\ i \neq j}}^{n} \gamma_{ij1} x_i \right\} \delta t, \delta x_j = \left\{ x_1 + \sum_{\substack{i=2 \\ i \neq j}}^{n} y_{ijj} x_i \right\} \delta t,$$

$$\delta x_k = \left\{ \sum_{\substack{i=2 \\ i \neq j}}^{n} \gamma_{ijk} x_i \right\} \delta t \qquad (k \neq 1, j). \tag{6}$$

For these ξ's, $\sigma(\xi)$ is unity to within an infinitesimal of the second order. Hence the increment to $\sigma(x)$ must vanish identically, so that

$$\gamma_{ij1} = \gamma_{ijj} = \gamma_{iji} = 0 \qquad (1, i, j \text{ distinct}), \tag{7}$$

$$c_k \gamma_{ijk} + c_i \gamma_{kji} = 0 \qquad (1, i, j, k \text{ distinct}). \tag{8}$$

By (7), (5) simplifies to

$$X_1 = x_1\xi_1 + \sum_{i=2}^{n} x_i \xi_i c_i, \; X_k = x_1\xi_k + x_k\xi_1 + \sum x_i\xi_j\gamma_{ijk} \; (k>1), \tag{9}$$

where, in the final sum, i and j range over distinct values from 2, ..., n, excluding k. This final sum is, therefore, absent if[4] $n=3$; whence $\sigma(X)$

has the term $2x_2\xi_2c_2.x_3\xi_3c_3$ which does not occur in $\sigma(x)\sigma(\xi)$. But $c_2c_3 \neq 0$ by hypothesis. Hence $n > 3$.

Hitherto we have not examined the conditions which follow from the final equations (3); these are

$$\gamma_{jik} = -\gamma_{ijk} \quad (i, j = 2, \ldots, n; i \neq j). \tag{10}$$

6. Taking $n = 4$ and applying (10), we see that (9) become

$$\begin{cases} X_1 = x_1\xi_1 + c_2x_2\xi_2 + c_3x_3\xi_3 + c_4x_4\xi_4, X_2 = x_1\xi_2 + x_2\xi_1 + \gamma_{342}(x_3\xi_4 - x_4\xi_3), \\ X_3 = x_1\xi_3 + x_3\xi_1 + \gamma_{243}(x_2\xi_4 - x_4\xi_1), X_4 = x_1\xi_4 + x_4\xi_1 + \gamma_{234}(x_2\xi_3 - x_3\xi_2). \end{cases} \tag{11}$$

These transformations do not in general form a group and hence are not generated by the corresponding infinitesimal transformations employed above. Hence it remains to require that $\sigma(X) = \sigma(x)\sigma(\xi)$ under the transformations (11). The conditions are seen to be

$$c_3c_4 = -c_2\gamma^2_{342}, c_2c_4 = -c_3\gamma^2_{243}, c_2c_3 = -c_4\gamma^2_{234}, c_4\gamma_{234} = c_2\gamma_{342} = -c_3\gamma_{243},$$

the first two of which reduce to the third by means of the last three equations. To these last can be reduced all the conditions (8) by means of (10).

Applying the transformation of variables which multiplies x_4, ξ_4, X_4 by γ_{234}, and leaves the remaining x_i, ξ_i, X_i unaltered, we get

$$\begin{cases} X_1 = x_1\xi_1 + c_2x_2\xi_2 + c_3x_3\xi_3 - c_2c_3x_4\xi_4, X_2 = x_1\xi_2 + x_2\xi_1 - c_3x_3\xi_4 + c_3x_4\xi_3, \\ X_3 = x_1\xi_3 + x_3\xi_1 + c_2x_2\xi_4 - c_2x_4\xi_2, \quad X_4 = x_1\xi_4 + x_4\xi_1 + x_2\xi_3 - x_3\xi_2. \end{cases} \tag{11^1}$$

These are the values obtained by Lagrange[5] in his generalization $\sigma(x)\sigma(\xi) = \sigma(X)$ of Euler's formula for the product of two sums of four squares. Then $x\xi = X$ gives the following multiplication table for the units:

$$\begin{cases} e_2^2 = c_2, e_3^2 = c_3, e_4^2 = -c_2c_3, e_2e_3 = e_4, e_3e_2 = -e_4, \\ e_2e_4 = c_2e_3, e_4e_2 = -c_2e_3, e_3e_4 = -c_3e_2, e_4e_3 = c_3e_2. \end{cases} \tag{12}$$

This algebra is associative and is the direct generalization of quaternions to a general field F which the writer[6] obtained elsewhere from assumptions including associativity. The four-rowed determinants of the general number x of this algebra equals $\sigma^2(x)$. The case $c_2 = c_3 = -1$ gives the algebra of quaternions, for which it is customary to write i, j, k instead of our units e_2, e_3, e_4.

7. It is not very laborious to show by the above method that the cases $n = 5$ and $n = 6$ are excluded. However, Hurwitz has proved that a relation of the form $\sigma(x)\sigma(\xi) = \sigma(X)$ is impossible if $n \neq 1, 2, 4, 8$. A slight simplification of his proof, together with an account of the history of this problem, has been given by the writer.[7] Hurwitz made no attempt to find all solutions when $n = 4$. We proceed to treat this problem.

Consider the case $c_i = -1$ to which the general case may be reduced by an irrational transformation. Then $\sigma(x) = \Sigma x_i^2$. We investigate the linear algebras having property (4), i. e.,

$$(x_1^2 + \ldots + x_n^2)(\xi_1^2 + \ldots + \xi_n^2) = X_1^2 + \ldots + X_n^2, \tag{13}$$

if

$$X_k = \sum_{j=1}^{n}\left(\sum_{i=1}^{n}\gamma_{ijk}\,x_i\right)\xi_j \qquad (k = 1, \ldots, n). \tag{14}$$

The matrix M of this substitution has the element $\Sigma_{i=1}^{i=n}\gamma_{ijk}x_i$ in the kth row and jth column. If this substitution is applied to a quadratic form in $X_1, \ldots, X_n$ of matrix Q, it is a standard theorem that we obtain a quadratic form in $\xi_1, \ldots, \xi_n$, whose matrix is $M'QM$, where M' is the transposed of M, being obtained from M by the interchange of its rows and columns. In our problem, Q is the identity matrix I whose elements are all zero except the diagonal elements which are 1. Hence, by (13),

$$M'M = (x_1^2 + \ldots + x_n^2)I. \tag{15}$$

When a homogeneous polynomial $\sigma(x_1, \ldots, x_n)$ of any degree has the property (4) of possessing a theorem of multiplication, the writer[8] has proved that we may apply a linear transformation on $x_1, \ldots, x_n$ which leave $\sigma(x)$ unaltered and one on $\xi_1, \ldots, \xi_n$ which leaves $\sigma(\xi)$ unaltered such that the new algebra has the principal unit e_1, so that γ_{1jk} and γ_{j1k} are both 0 if $j \neq k$, and both unity if $j = k$.

Hence $M = x_1M_1 + \ldots + x_nM_n$, where γ_{ijk} is the element in the kth row and jth column of M_i, whence $M_i = I$. Thus (15) gives

$$M_i' = -M_i,\ M_i'M_i = I,\ M'_iM_j + M_j'M_i = 0 \quad (i>1, j>1, j \neq i). \tag{16}$$

In view of the values of γ_{jik}, and $M_i^1 = -M_i$, we have, when $n = 4$,

$$M_2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -\gamma_{223} & -\gamma_{224} \\ 0 & \gamma_{223} & 0 & -\gamma_{234} \\ 0 & \gamma_{224} & \gamma_{234} & 0 \end{pmatrix}, \qquad M_3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & -\gamma_{323} & -\gamma_{324} \\ 1 & \gamma_{323} & 0 & -\gamma_{334} \\ 0 & \gamma_{324} & \gamma_{334} & 0 \end{pmatrix},$$

$$M_4 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -\gamma_{423} & -\gamma_{424} \\ 0 & \gamma_{423} & 0 & -\gamma_{434} \\ 1 & \gamma_{424} & \gamma_{434} & 0 \end{pmatrix}.$$

By $M_i'M_i = I$, we have $M^{2i} = -I$, which gives

$$\gamma_{223} = \gamma_{224} = \gamma_{323} = \gamma_{334} = \gamma_{424} = \gamma_{434} = 0,\ \gamma^2 = \delta^2 = \epsilon^2 = 1,$$

where $\gamma = \gamma_{234}$, $\delta = \gamma_{324}$, $\epsilon = \gamma_{423}$. Hence

$$M_2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\gamma \\ 0 & 0 & \gamma & 0 \end{pmatrix},\quad M_3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\delta \\ 1 & 0 & 0 & 0 \\ 0 & \delta & 0 & 0 \end{pmatrix},\quad M_4 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -\epsilon & 0 \\ 0 & \epsilon & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{17}$$

The final condition (16) states that M_iM_j is skew-symmetric. The products M_2M_3, M_2M_4, M_3M_4 of matrices (17) are seen at once to be skew-symmetric if and only if $\delta = -\gamma$, $\epsilon = \gamma$, and then $M_2M_3 = \gamma M_4$, $M_2M_4 = -\gamma M_3$, $M_3M_4 = \gamma M_2$. Writing i, j, k for M_2, M_3, γM_4, we have the multiplication table of quaternions. Or we may form the matrix M and write X_k for the sum of the products of the elements of its kth row by $\xi_1, \ldots, \xi_4$, and take $\gamma = 1$ (by multiplying x_4, ξ_4, X_4 by γ); we obtain (11′) for $c_2 = c_3 = -1$. Hence we have again obtained the quaternion algebra without assuming the associative law. The case $n = 8$ is being investigated in this way by one of my students.

[1] Frobenius, *Jour. für Math.*, **84**, 1878 (59).

[2] Dickson, *Linear Algebras*, "Cambridge Tracts in Mathematics and Mathematical Physics," No. **16**, 1914 (10–12).

[3] Dickson, *Bull. Amer. Math. Soc.*, **22**, 1915 (53–61).

[4] By (7), $e_2e_3 = e_3e_2 = 0$.

[5] Lagrange, *Nouv. Mém. Acad. Roy. Sc. de Berlin*, année 1770, Berlin, 1772 (123–133); *Oeuvres* de Lagrange, **3**, 1869 (189). Reproduced in Dickson's *History of the Theory of Numbers*, II, 1920 (279–281).

[6] Dickson, *Trans. Amer. Math. Soc.*, **13**, 1912 (65).

[7] Dickson, *Annals of Math.*, **20**, 1919 (155–171, 297).

[8] Dickson, *Comptes Rendus du Congrès Internat. Math.*, Strasbourg, 1920 (131–146).

NOVOCAINE AS A SUBSTITUTE FOR CURARE[1]

BY JOHN F. FULTON, JR.

HARVARD UNIVERSITY

Communicated by G. H. Parker, March 3, 1921

Since the recent war, the need of a substitute for the Indian arrow poison, curare, has been keenly felt in many physiological laboratories. While investigating the activity of certain local anesthetics, it was found that novocaine, in its effect upon the neuro-muscular mechanism of frogs, duplicates in many particulars the unique action of curare.

If the sciatic nerve of a sciatic-gastrocnemius preparation is bathed in a strong solution of novocaine (2.5 per cent in water or in physiological salt solution) for as long as twenty minutes, no decrease in its conductivity can be observed. However, if the muscle itself is bathed in such a solution (by direct immersion or, "painting" with a camel's hair brush) the power of reacting to nervous stimulation is destroyed within three to five minutes, though ability to respond by contraction to direct electrical stimulation remains unimpaired. Thus, in the action of novocaine there is a complete duplication of the properties originally described by Claude Bernard for curare.

Whether novocaine acts directly upon the end-plates of the motor fibers or upon some membrane intermediate between the plates and the muscle has not been determined. A dye that I have made by linking novocaine with a benzene nucleus was found to be physiologically active, like the 2.5% novocaine, and appears to stain the elements acted upon by the novocaine. When the stain is diazotized directly into the living muscle, by putting the tissue first into novocaine and then into a solution of the staining base, only the muscle nuclei take the stain deeply, the nervous elements of the end-plates as well as the motor fibers remaining uncolored. It seems reasonably certain, therefore, that novocaine acts upon some constituent of the neuro-muscular mechanism beyond the end-plates. The significance of the affinity of the dye for the muscle nuclei is as yet unknown. The object of this note is to direct the attention of physiologists to a convenient substitute for curare.

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[1] Contributions from the Zoölogical Laboratory of the Museum of Comparative Zoölogy at Harvard College. No. 330.

NOTE ON A METHOD OF DETERMINING THE DISTRIBUTION OF PORE SIZES IN A POROUS MATERIAL

BY EDWARD W. WASHBURN

DEPARTMENT OF CERAMIC ENGINEERING, UNIVERSITY OF ILLINOIS

Communicated by W. A. Noyes, February 12, 1921

The pressure required to force mercury into a capillary pore of radius, r, is $\frac{-2\gamma\cos\theta}{r}$, where γ is the surface tension and θ the angle of contact. Upon this relation can be based a method for determining the effective pore diameters in a porous material such as charcoal. If pores of various diameters are present, one may determine also the fraction of the total porosity which is due to pores having effective diameters lying between any two stated limits. The procedure would be as follows:

The coarsely granular sample of the thoroughly outgassed material is weighed and placed in a steel pressure bomb which is then evacuated until all adsorbed gases are removed. Pure mercury is then admitted to fill the bomb and a series of pressure and volume measurements are made at various pressures up to the highest pressure it is desired to employ.

The decrease in volume, ΔV, accompanying a small pressure increase of Δp, in any part of the range must evidently be due to the filling of pores whose effective radii lie between the limits r and $r - \Delta r$, or

$$\frac{\Delta r}{\Delta p} = \frac{-2\gamma\cos\theta}{p^2}.$$

A blank experiment without the porous material should of course be made in order to correct for the compressibility of the mercury and the expansion of the bomb under pressure. For accurate results the compressibility of the porous material, and the variation of γ and θ with p should also be known.

The value of θ could be determined from an X-ray photograph of a mercury meniscus in a capillary composed of the material under examination, or by the drop-shape method.

AN APPARENT CASE OF NON-MENDELIAN INHERITANCE IN DATURA DUE TO A DISEASE

By Albert F. Blakeslee

Station for Experimental Evolution, Cold Spring Harbor, N. Y.

Communicated by C. B. Davenport, February 26, 1921

In the common Jimson Weed (*Datura Stramonium*), spiny or armed capsules are dominant to smooth or *inermis* capsules. A cross between two plants, each homozygous for a single member of this pair of characters will produce spiny fruited offspring in the F_1 and a ratio of 3 spiny to 1 smooth plant in the F_2 generation.

In 1915, the writer found a single *inermis* plant in a culture where smooth capsules could not have occurred through segregation. It was considered a new *inermis* mutation and its inheritance was therefore studied in crosses with normals.

The new form was called Quercina on account of the increased oak-like dentation of its leaves. The most conspicuous character on the mature plant was the partial or complete suppression of spines on the capsules. An examination of the plants throughout the growing and flowering condition indicated that other parts of the plant were also involved and showed such changes as the slitting of the normally undivided corolla, the absence of pollen, which caused the plant to depend upon outcrossing in order to set seed, and certain other characters associated with less vigorous growth.

Later investigation showed that the Quercina character occurred spontaneously in the garden cultures in many ways like a vegetative mutation. In a single year's test, about $1^1/_4$ per cent of the normal plants in the field took on the Quercina character by the last of the season. This Quercina character generally shows itself weakly in a single branch and gradually spreads to all the new growth. It occasionally happens on plants which are acquiring the Quercina character that capsules will be found in a transitional condition with their spines only partially reduced. Sometimes some of the valves may be entirely smooth and others on the same capsule fully spined.

So far as we can judge from the literature, other investigators who have worked on the Jimson Weed found have Quercina plants in their cultures. They seem not to have noticed, however, any of the distinctive characters other than those shown by the capsules.

Naudin[1] reports in an F_1 between an *inermis* variety (*D. laevis*) of *D. Stramonium* and the armed type, that while most of the plants had spiny fruits, others had fruits with reduced spines. Many of the capsules on three out of the forty plants in this generation were very spiny on part of the surface while totally smooth on the rest. Naudin believed that they united thus by distinct and separate compartments the distinctive traits of the parental types *D. laevis* and *D. Stramonium.* He calls this "*hybridité disjointe*" and cites in this connection the condition in the graft chimera *Cytisus Adami.* In a series of F_2 plants from the same original cross Naudin found 6 individuals out of 38 which again showed more or less well marked his "*hybridité disjointe*" and which were presumably Quercinas.

Godron[2] reports finding capsules partly spiny and partly smooth. He objects, however, to Naudin's interpretation that the separation of the fruit into smooth and spiny portions is due to their origin from smooth and spiny parents, since he says that he has found this condition when *both* parents had spines.

Bateson and Saunders,[3] after discussing the intermediate color of the flowers in the F_1 between white and purple flowered forms, say: "The occurrence of intermediate forms was also occasionally noticeable in the fruits. Among the large number of capsules examined, there were some of the mosaic type, in which part of the capsule was prickly and the remainder smooth, while others suggesting a blend, were more or less prickly all over, but the prickles were much reduced in size, and often formed mere tubercles. These mosaics occurred as rareties both on prickly individuals and on smooth ones still more rarely." Further evidence pointing to the Quercina nature of these abnormal capsules is given in their following statement: "Such intermediate fruits were most often found towards the end of the flowering season."

It may be stated that the writer has grown many thousand plants of the Jimson Weed, including large numbers of individuals that were heterozygous for *inermis* capsules, but has never observed any except Quercinas which showed a mosaic arrangement of the spines on the fruits, with some valves smooth and others more or less spiny. Despite their misconception of the nature of the phenomenon which they observed, it is highly probable that the investigators mentioned were describing the same type of plant that we have considered under the term Quercina.

The Quercina character does not breed true. It is transmitted by seed to about 79 per cent of its offspring when pollinated from normal plants. The remaining 21 per cent normal offspring do not produce Quercina seedlings in the next generation, and hence do not carry the character in a recessive condition.

Two plants which were changing over from a normal to a Quercina condition were investigated with regard to the type of offspring obtained from their individual capsules. There seems to be only a rough correspondence between the strength of the Quercina character in the parent and the number of Quercina plants in its offspring.

As has been stated already, Quercina flowers are characteristically devoid of pollen. In a few plants, however, which were becoming transformed into Quercinas, pollen was obtained and used in crosses onto normal plants. Quercina offspring were obtained from such crosses showing that the character in question can be transmitted through the male.

The non-Mendelian nature of the breeding results with Quercina plants as well as the spontaneous occurrence of the character on branches of normal plants and its spread to the new growth suggested the desirability of attempting to transmit the character by grafting. This attempt was entirely successful and Quercina cions grafted onto normal plants of the Jimson Weed causes the new growth of the stock to take on the appearance of Quercina. There is ample evidence, therefore, to indicate that the cause of the Quercina complex, which by ourselves as well as by earlier investigators had been supposed to be a manifestation of a genetic character, is in fact a disease transmissible by grafting.

It has not been found possible to infect plants artificially by rubbing with diseased leaves nor by inoculation of expressed juice from Quercina plants.

The profound morphological changes brought about in the leaves and especially in the flowers and fruit are such that Quercina individuals would be considered worthy of specific if not of generic separation if 100 per cent of the seedlings instead of only 79 per cent came true to the Quercina complex. As the facts stand, however, there is much in the behavior of Quercina plants which suggests genetic phenomena.

To one who has read the data presented in the foregoing pages, certain similarities may be apparent between the Quercina in the Jimson Weed and rogues in peas investigated by Bateson and Pellew.[4] It will not be desirable, however, to discuss the two forms more in detail until grafting experiments with the rogues in peas have given definite results.

A detailed account of the Quercina disease with photographs will appear shortly in the *Journal of Genetics.*

[1] Naudin, Ch., Nouvelles Recherches sur L'Hybridité dans les Végétaux, *Nouv. Arch. Mus.*, **1,** 1865 (41–54).

[2] Godron, D. A., *Des hybrides et des metis de Datura.*, Nancy, 1873 (1–75).

[3] Bateson, W. and Saunders, E. R., Report to Evolution Committee of the Royal Society, **1,** 1902 (21–32).

[4] Bateson, W. and Pellew, Caroline, "The Genetics of 'Rogues' among Culinary Peas," *Proc. Royal Soc.*, **91,** 1920 (186–195).

THE EFFECT OF A VARIATION IN MILLING ON THE DIGESTIBILITY OF WHEAT FLOURS

By C. F. Langworthy and Arthur D. Holmes

Contribution from the Office of Home Economics, U. S. Department of Agriculture

Communicated by W. A. Noyes, February 16, 1921

Introduction.—Cereals are almost universally the great source of carbohydrate food in the human ration, and with moderate additions of fat and protein they are capable of sustaining life and health over long periods of time. They may be stored for years with little deterioration in quality, are easily transported and yield the greatest amount of food for the labor expended in producing them. Of the cereal grains, wheat, largely because of its gluten content which causes it to produce superior bread making flour, has come to be considered as an almost indispensable article of diet throughout Western Europe and America. The normal, annual per capita consumption of wheat in this country is approximately five bushels.[1] Valuable as wheat is in the human dietary in normal times, it is even more important in times of stress, for a disturbance in the bread supply has an immediate and often harmful effect on national morale.

The shortage in the food supply of the world in 1917 made the collection and correlation of data concerning human nutrition of utmost importance and many matters were re-examined in the hope of providing additional data of value. Extensive investigations of the chemical composition, digestibility, and nutritive value of wheat and wheat products have been made by European investigators and in the United States. Earlier investigations of this Office, undertaken with the coöperation of the Maine and Minnesota Agricultural Experiment Stations, included studies of the digestibility of wheat flours prepared under a variety of milling conditions from wheat grown under uniform climatic, seasonal and soil conditions. The results[2] obtained showed that the protein of white flour (72% extraction) was 88.1% digested, that of entire wheat flour (85% extraction) was 81.9% digested, and that of graham flour (100% extraction) was 76.9% digested. While the data were sufficient to permit general conclusions regarding the effect of variation of milling on digestibility, the unprecedented world shortage of food during the World War made rigid economy in the use of food materials necessary and in 1917 the U. S. Food Administration requested this Office to secure additional data concerning the relative digestibility of flours representing a series of extractions from the same lot of wheat. The results obtained from the work thus undertaken were available to the Food Administration during the war period. The findings, however, are not limited in their application to special conditions under which the work was carried out but are of equal interest in the consideration of general problems of nutrition and the economic utilization of the food supply under normal conditions.

The flours studied were prepared from the wheat mixture furnished to the flour millers by the U. S. Grain Corporation late in 1917. It consisted of the following:

20%	choice hard spring (largely Marquis)
25%	58 lbs. spring
15%	velvet chaff
25%	slightly smutty spring
10%	durum
5%	Kansas and Oklahoma

At the request of the Food Administration the wheat was so milled by commercial flour mills that a uniform series of extractions was obtained. As required by the U. S. Food Administration regulations then in force, the amount of wheat used was limited in the proportion of 264 lbs. of 58-lb. wheat for 196 lbs. of flour. The milling data follow:

75%	out of	72%	extraction yielding	54%	of the wheat
95%	out of	74%	extraction yielding	70%	of the wheat
100%	out of	85%	extraction yielding	85%	of the wheat
100%	out of	100%	extraction yielding	100%	of the wheat

For the sake of convenience the above flours will be referred to according to the percentage of the wheat extracted. Described in ordinary commercial terms, the 54% flour contained only patent flour. The 70% extraction, that is 95% of 74% extraction, is what is known commercially as a 95% patent, or "standard patent" and contained patent, 1st clear, and a small portion of the 2nd clear. The 85% flour corresponds to the old-fashioned "whole-wheat" flour, and contained patent, 1st clear, 2nd clear, Red Dog, and Shorts. The 100% flour corresponds to that known formerly as graham flour and contained the entire wheat kernel.

These experiments were begun before the Food Administration had adopted the milling of what was later known as "war flour." The latter was sometimes known as "100% war flour," but this did not mean that it contained 100% of the wheat. In milling the "war flour," 100 lbs. of cleaned wheat was separated into two products: about 75 lbs. of flour (never less than 74.3 lbs.) and about 25 lbs. of feed, giving a flour of about 75% extraction. The 95% patent or 70% extraction flour used in these experiments was similar to the "war flour" save that 5% of low grade and 2nd clear stock had been removed to improve its bread-making quality.

Feeding experiments were conducted with healthy young men according to the method commonly used by the Office of Home Economics in studies of digestibility.[3]

In order to ascertain whether the subjects acquired during the period of the experiments any increased tolerance for the coarser flours the squad was divided into groups of five subjects each, and one group was served the 54%, 70%, 85%, and 100% flours in the order given while the second group was served the flours in the reverse order. The results of the 139 digestion experiments reported below do not indicate that the first group of subjects acquired any tolerance for the coarser flours not possessed by the second group which was served the coarser flours first.

The experimental diet was so chosen that its preparation should involve a minimum of labor, that the bread should supply the larger portion of the total protein, and that it should be representative of a simple mixed diet. The diet consisted of bread, fruit (orange), butter, sugar, with tea or coffee as a beverage if desired.

The fruit, butter, and sugar were served as purchased from a nearby market. A quantity of bread sufficient to supply all the subjects for one day was prepared daily, the ingredients being used in the following proportion:

3 cups sifted flour
1 teaspoon salt
1 tablespoon sugar
1/8 to 1/2 cake yeast
1 tablespoon hydrogenated vegetable oil
1 cup water

While data relative to the digestibility of the diet as a whole are of value, interest was primarily centered in the digestibility of the bread or rather the flour from which the bread was made. The digestibility of the protein and carbohydrate supplied by the different breads has been estimated by making a correction for the undigested protein and carbohydrate resulting from the accessory foods by a method outlined in detail in earlier publications.[4] It has been assumed from the results of the early investigations of this Office[5] that the digestibility of the protein of butter is 97% and protein of fruit, 85%; of the carbohydrate of fruit, 90%; and of sugar, 98%.

In planning the investigation, it was decided to make a sufficient number of digestion experiments so that the average results obtained should be of general application. The experimental periods were to be long enough so that the effect of any irregularity in the rate of passage of food residues through the alimentary tract, and any error in the separation or collection of feces would be practically negligible. The experimental periods which were from 15 to 25 days in length were subdivided and considered as separate three-day experiments, following one directly after the other.

The subjects for the experiments were students in local educational institutions, and were selected with care in order that the conclusions might be applicable to the average normal adult. An attempt was made to include in the squad some accustomed to strenuous exercise, some accustomed to take light exercise, and some who took little recreation, in order to have under observation individuals whose peristaltic action was normal and regular and others who had a tendency toward constipation.

Summary of Results.—The results obtained in the studies of 54%, 70%, 85%, and 100% flours are summarized in the table which follows:

TABLE 5

SUMMARY OF EXPERIMENTS ON THE DIGESTIBILITY OF WHEAT FLOURS

KIND OF FLOUR USED IN PREPARATION OF BREAD	NUMBER OF EXPTS.	AVERAGE AMOUNT OF BREAD EATEN PER MAN PER DAY	DIGESTIBILITY OF ENTIRE RATION			DIGESTIBILITY OF BREAD PROTEIN	DIGESTIBILITY OF BREAD CARBOHYDRATE
			PROTEIN	FAT	CARBOHYDRATE		
		GRAMS	PER CENT	PER CENT	PER CENT	PER CENT	PER CENT
54% flour	43	600	87.8	96.5	98.8	87.7	99.7
70% flour	42	564	90.1	96.1	99.0	90.1	99.9
85% flour	21	472	87.1	96.9	97.5	87.1	98.5
100% flour	33	663	84.2	93.9	95.0	84.2	94.4

In the above experiments the subjects ate on an average considerably more than a pound of bread daily for periods of from 6 to 20 days without producing any digestive disturbances, which indicates that wheat flours, regardless of percentage of extraction, are well tolerated by the human body.

The coefficients of digestibility, obtained from the 139 digestion experiments reported above, 87.7% for protein and 99.7% for carbohydrate of 54% flour; 90.1% for protein and 99.9% for carbohydrate of 70% flour (95% patent); 87.1% for protein and 98.5% for carbohydrate of 85% ("whole wheat") flour; and 84.2% for protein and 94.4% for carbohydrate of 100% flour, show these flours to be well digested. From these results it appears that the U. S. Food Administration in attempting to obtain efficient utilization of the wheat supply secured did well to specify that wheat should be milled at 75% extraction, that is an extraction similar to the one which in these experiments showed the highest proportion of digestible nutrients.

The digestibility of the fat content of the experimental diet was quite uniform and was practically identical with that of butter and "shortening"[6] which comprised the major portion of the fat consumed, except in the series of experiments with 100% flour in which the fat was only 93.7% digested.

Attention was given to the effect of the different flours on peristaltic action. The 54% and 70% flours did not tend to produce constipation during a period of 15 to 18 days, and although a somewhat freer movement of the bowels resulted from the continued use of 85% and 100% flours no case of pronounced laxative effect was noted.

In general the results of the digestion experiments here reported are in accord with conclusions drawn from earlier studies of the digestibility of wheat flours. The digestibility of the 70% (95% patent) flour was the highest, that of the 54% flour was slightly greater than that of the 85% ("whole wheat") flour, while the digestibility of the 100% (graham) flour was lowest of all those studied. Since the flavor of bread made with the different flours varies, the use of different kinds for bread making is an easy way of giving variety to the diet.

[1] U. S. Dept. Agr., Bur. Crop Estimates, *Monthly Crop Rept.*, **3,** 1917, No. 10 (99).

[2] U. S. Dept. Agr., *Office Expt. Sta. Bull.* **85,** 1900 (32–33); *Bull.* **101,** 1901 (33); *Bull.* **126,** 1903 (29, 45); *Bull.* **143,** 1904 (32); *Bull.* **156,** 1905 (36).

[3] U. S. Dept. Agr., *Bull.* **310,** 1915; **617,** 1919; **717,** 1919.

[4] U. S. Dept. Agr., *Bull.* **470,** 1916 (7); **525,** 1917 (4).

[5] *Connecticut Storrs Sta. Rpt.*, 1899 (104).

[6] U. S. Dept. Agr., *Bull.* **310,** 1915.

THE MATHEMATICAL THEORY OF THE APPORTIONMENT OF REPRESENTATIVES[1]

By Edward V. Huntington

Harvard University, Cambridge, Mass.

Communicated by E. H. Moore, February 14, 1921

The Problem.—The exact quota to which each state is theoretically entitled on the basis of population usually involves a fraction. *The problem is, to replace these exact quotas by whole numbers* in such a way that the resulting injustice (due to adjustment of the fractions) shall be as small as possible.

This problem has been the subject of violent debate in Congress for the past one hundred years, a new method of apportionment having been proposed after almost every decennial census. None of these methods, however, possesses any satisfactory mathematical justification. The need of a strictly mathematical treatment of the problem having been called to the writer's attention by Dr. J. A. Hill, Chief Statistician of the Bureau of the Census, the following solution has been worked out on the basis of two very simple postulates. The new method may be called the *Method of Equal Proportions.*

Let N be the total number of representatives, A, B, C, ... the populations of the several states, and a, b, c, ... the number of representatives assigned to each.

Fundamental Principle.—In a satisfactory apportionment between two states (A greater than B), we shall agree that A/a and B/b should be as nearly equal as possible; also a/A and b/B; also A/B and a/b; also B/A and b/a.

Now to say that two quantities are "nearly equal" may be interpreted to mean: *either*, that the difference between the quantities is nearly zero; *or*, that the ratio between them is nearly one.

(Here the difference "between" two quantities means the larger minus the smaller. Similarly, the ratio "between" two quantities means the larger divided by the smaller.)

If we adopt the "difference" interpretation, we have:

POSTULATE I*a*. *The difference between* A/a *and* B/b; or

POSTULATE I*b*. *The difference between* a/A *and* b/B; or

POSTULATE I*c*. *The difference between* A/B *and* a/b; or

POSTULATE I*d*. *The difference between* B/A *and* b/a; *should be as near zero as possible.*

If, on the other hand, we adopt the "ratio" interpretation, we have:

POSTULATE I. *The ratio between* A/a *and* B/b (*or the ratio between* a/A *and* b/B; *or the ratio between* A/B *and* a/b; *or the ratio between* B/A *and* b/a; *all of which have the same value*) *should be as near unity as possible.*

Since there is no way of choosing, mathematically, between Postulates I*a* and I*b* or between Postulates I*c* and I*d*, and since these four demands lead to four different results, we shall reject all four of them and *adopt Postulate I as the proper interpretation of our Fundamental Principle.*

The case of two states is thus disposed of.

For the case of three or more states, one further principle is required, which we state as follows:

POSTULATE II. *In a satisfactory apportionment, there should be no pair of states which is capable of being "improved" by a transfer of representatives within that pair*—the word "improvement" being understood in the sense implied by the test already adopted for the case of two states, and the rare cases of "no choice" being decided in favor of the larger state.

From these two postulates the following theorem can then be deduced:

THEOREM I. *For any given values of* A, B, C, ... *and* N, *there will always be one and only one satisfactory apportionment in the sense defined by Postulates I and II.* No further principles are required.

A working rule for computing this "best" apportionment in any given case is found to be as follows:

Working Rule.—Multiply the population of each state by as many of the numbers

$$\text{Inf.},\ 1/\sqrt{1 \times 2},\ 1/\sqrt{2 \times 3},\ 1/\sqrt{3 \times 4},\ \ldots$$

as may be necessary, and record each result, together with the name of the state, on a small card. Arrange these cards according to the magnitude of the numbers recorded upon them, from the largest to the smallest, thus forming a *priority list* for the given states (the cards marked "Inf." being placed at the head of the list, arranged among themselves in order of magnitude of the populations of the states). Finally, *assign the representatives, from the 1st to the Nth, to the several states in the order in which the names of the states occur in this priority list.* (It should be noted that this method satisfies automatically the constitutional requirement that every state shall have at least one representative.)

This method may be called the "*method of the geometric mean,*" since the "multipliers" are the reciprocals of the geometric means of consecutive integers.

The solution of the problem is thus complete.

Alternative Methods.—If we had adopted Postulate Ia or Ib we should have been led, in like manner, to two other methods which may be called the *method of the harmonic mean* (Ia), and the *method of the arithmetic mean* (Ib), since the "multipliers" in the working rules are as follows:

$$\begin{array}{llllll} (\mathrm{I}a) & \text{Inf.}, & \dfrac{1+2}{2(1\times 2)}, & \dfrac{2+3}{2(2\times 3)}, & \dfrac{3+4}{2(3\times 4)}, & \cdots \\ (\mathrm{I}b) & 2, & 2/3, & 2/5, & 2/7, & \ldots \end{array}$$

It can be shown that method Ia favors the smaller states more than method I does, while method Ib favors the larger states more than method I does. Since there is no mathematical reason for adopting either of the two Postulates Ia and Ib to the exclusion of the other, both should be rejected.

Postulates Ic and Id also determine two distinct methods, which may be called *the two methods of similarity ratios.* It can be shown that Ic favors the small states even more than Ia does, while Id favors the large states even more than Ib does, so that both should be rejected.

Each of these four methods violates three of the four conditions expressed in our Fundamental Principle, while the method of the geometric mean satisfies all these conditions simultaneously.

The following further methods are suggested by the Theory of Least Squares.

In a theoretically perfect apportionment, A/a would be equal to P/N, and a/A to N/P (where P is the total population). Hence, in place of Postulates I and II, we might consider the following:

POSTULATE IIIa. *The sum of the squares of the deviations of the* A/a *from their true values;* or

POSTULATE IIIb. *The sum of the squares of the deviations of the* a/A *from their true values; should be a minimum.*

It can be shown, however, that IIIa favors the smaller states even more than Ic does, while IIIb favors the larger states even more than Id does. In other words, Postulates IIIa and IIIb violate, in opposite directions, all four of the conditions expressed in our Fundamental Principle. Since there is no mathematical reason for adopting either to the exclusion of the other, both should be rejected.

The same remark applies if in Postulates III*a* and III*b* the word "square" is replaced by "absolute value."[2]

Hence it is clear that if a *numerical measure of injustice* is desired, both the deviation of A/a and the deviation of a/A should be taken into account together. That is, any formula which reports A/a, say, as too large by a certain amount, should also report a/A as too small by the same amount. The formulas suggested by the simple application of the idea of least squares, as shown in the preceding paragraph, do not have this property. A combination of these formulas suggests, however, the following postulate, in which α denotes the theoretical value of a.

POSTULATE III. *In a satisfactory apportionment, the sum* T *of terms of the form* Ae^2 *where*

$$e = \frac{(a/A) - (\alpha/A)}{\sqrt{(a/A)(\alpha/A)}} = \frac{(A/\alpha) - (A/a)}{\sqrt{(A/\alpha)(A/a)}} = \frac{a - \alpha}{\sqrt{a\alpha}}$$

should be a minimum.[3]

For purposes of computation, this total error, T, may be replaced by an average error, $E = \sqrt{S/N}$, where S is the sum of terms of the form $\alpha e^2 - (a - \alpha)^2/a$.

The method determined by Postulate III is precisely the same as the method of equal proportions based on Postulates I and II.

[1] This article contains the substance of two papers presented to the American Mathematical Society, December 28, 1920, and February 26, 1921. Further details, with proofs and examples, will be published either in the Transactions of the American Mathematical Society, or in the Quarterly Publication of the American Statistical Association, or in the American Mathematical Monthly.

For the history of the subject see W. F. Willcox, "The Apportionment of Representatives" (presidential address at the annual meeting of the American Economic Association, December 1915), published in the American Economic Review, Vol. 6, No. 1, Supplement, pp. 1–16, March, 1916. See also 62d Congress, 1st Session, House of Representatives, Report No. 12, pp. 1–108, April 25, 1911, and John H. Humphreys, "Proportional Representation," London, 1911.

The most important of the methods hitherto known are four:

The *Vinton method of 1850*, long in use in Congress, is known to lead to an "Alabama paradox;" that is, an increase in the total size of the House may cause a decrease in the representation of some state.

The *Hill method of alternate ratios*, proposed by Dr. J. A. Hill in 1910 but not adopted, comes very near to satisfying the postulates of the present paper, and uses for the first time (though only partially) the idea of the geometric mean. The method is incomplete however, since it can be shown to lead to an Alabama paradox.

The *Willcox method of major fractions*, devised by Professor W. F. Willcox in 1900–1910, and now in use in Congress, employs, in effect, a working rule like ours with multipliers: Inf., 2/3, 2/5, 2/7, . . . ; it is essentially the same as the method of the arithmetic mean, and therefore favors the larger states unduly, just as the hitherto unsuspected but equally justifiable method of the harmonic mean favors the smaller states unduly. (It may be noted that the name "major fractions" is somewhat misleading, since the Willcox major fraction is not a major fraction of the true quota, but a major fraction of an artificial quota, scaled up or down from the true quota to meet the requirements of the computation.)

The *d'Hondt method,* originated in Belgium and now widely used in European elections, employs "multipliers" 1, 1/2, 1/3, 1/4... and can be shown to favor the larger states to the extent of violating all four of the conditions expressed in our Fundamental Principle.

[2] If one should try to minimize the *sum of the squares* (or the sum of the absolute values) *of the deviations of the a's themselves* from their true values (with or without "weighting" by the population of the state), the resulting methods would all lead to an Alabama paradox. The same is true of the weighted sum of the absolute values of the deviations of a/A (or of A/a). The same is also true of the absolute values of the logarithms of the ratios between the a'S and their true values.

[3]This Postulate III was added on April 23, after Professor F. W. Owens had shown (at the meeting of the American Mathematical Society on February 26) that the method of minimizing the sum of terms like $A[(a/A) - (\alpha/A)]^2$ leads to th same result as the Willcox method of major fractions. It may be noted that the method of minimizing the sum of terms like $a[(A/a) - (A/\alpha)]^2$ leads, not as one might expect, to the method of the harmonic mean, but to the method of the geometric mean.

CURRENT MAPS OF THE LOCATION OF THE MUTANT GENES OF DROSOPHILA MELANOGASTER[1]

BY CALVIN B. BRIDGES

COLUMBIA UNIVERSITY, N. Y.

Communicated by T. H. Morgan, December 5, 1920

The maps that have been published[2] showing the distribution of the mutant genes of D. melanogaster can now be much improved because of the discovery of new mutants and the accumulation of crossover data. Figure 1 gives in simplified form the maps that are in use in our laboratory.

The distances on the maps are based on the total amount of crossing over between the loci, one unit of distance representing one per cent of crossing over. The map-distances are the same as the observed crossover values or "percentages of exchange" whenever the two loci considered are so close together that no, or only a negligible amount of, double crossing over occurs between them. In the first (X-) chromosome this practical equivalence of map-distance and exchange-value holds for loci not farther apart than about 15 units. In the middle of the second chromosome and of the third chromosome the equivalence holds for only about 10 units. In the end-regions of the second and third chromosomes it holds up to nearly 20 units. For distances somewhat greater than these the map-distances exceed the observed percentages of exchange by an amount equal to twice the percentage of double crossing over between the given loci. For still greater distances the difference includes also three times the percentage of triples. The number of quadruple crossovers is negligible except perhaps when the whole length of the second chromosome is to be considered. For each chromosome and for each region within a chromosome the amount of this multiple crossing over is characteristic, and may not be the same in amount for different sections of equal map-distance. Because of this variation, the accurate expression of the relation between map-distance and exchange-values for the more distant loci, requires a table of conversion corrections for each pair of loci considered. Such tables will be published with the more detailed maps of the chromosomes. In general, the correction is relatively slight with distances that are under 20–40, but for longer intervals the correction increases at an accelerated rate. We have not met with percentages of exchange that exceed 50.0, though two of the maps are about a hundred units long.

The map of the third chromosome is the most accurate, since the calculation of the distances between the principal loci is made on the basis of all data up to 1920, and an improved method of weighting and interrelating the data has been followed. Relatively little change in these primary distances—the "triangulation" of the map—is expected with the further accumulation of data. There is still some uncertainty with regard to the region to the left of spineless, for the different sets of data upon that region may not be comparable because of the possible presence of crossover variations. The data used in the calculations for the primary distances in the first and second chromosomes are homogeneous, and although not including the last four years' work, are still fairly ample in amount. Changes are to be expected when these two maps are recalculated with the improved method and complete data.

The most useful mutants are those that are separable from the wild-type with completeness and ease, that are not inferior to the wild-type in viability and productivity, and that do not interfere with the use in the same experiment of any large class of the other mutant characters. Dominants are more valuable than recessives. Mutants accurately located in the chromosome are more valuable than those whose positions are less well established, though if the other desirable features mentioned above are present in a given new mutant the position will be found rather quickly. A very large factor in the value of the mutant is its position in the chromosome. The end positions are most valuable. Throughout the remainder of the chromosome the most favorable positions are those evenly spaced and just close enough together so that no double crossing over occurs between them. If the interval is too small there is trouble in getting double recessives, and the crossover classes are so small that large totals are required to make differences significant.

The mutants that fulfill all the above requirements most nearly are followed by an asterisk (*) in the maps. Their loci are the primary bases with relation to which the other mutants are located. There are several cases in which two or more excellent mutants affecting quite different

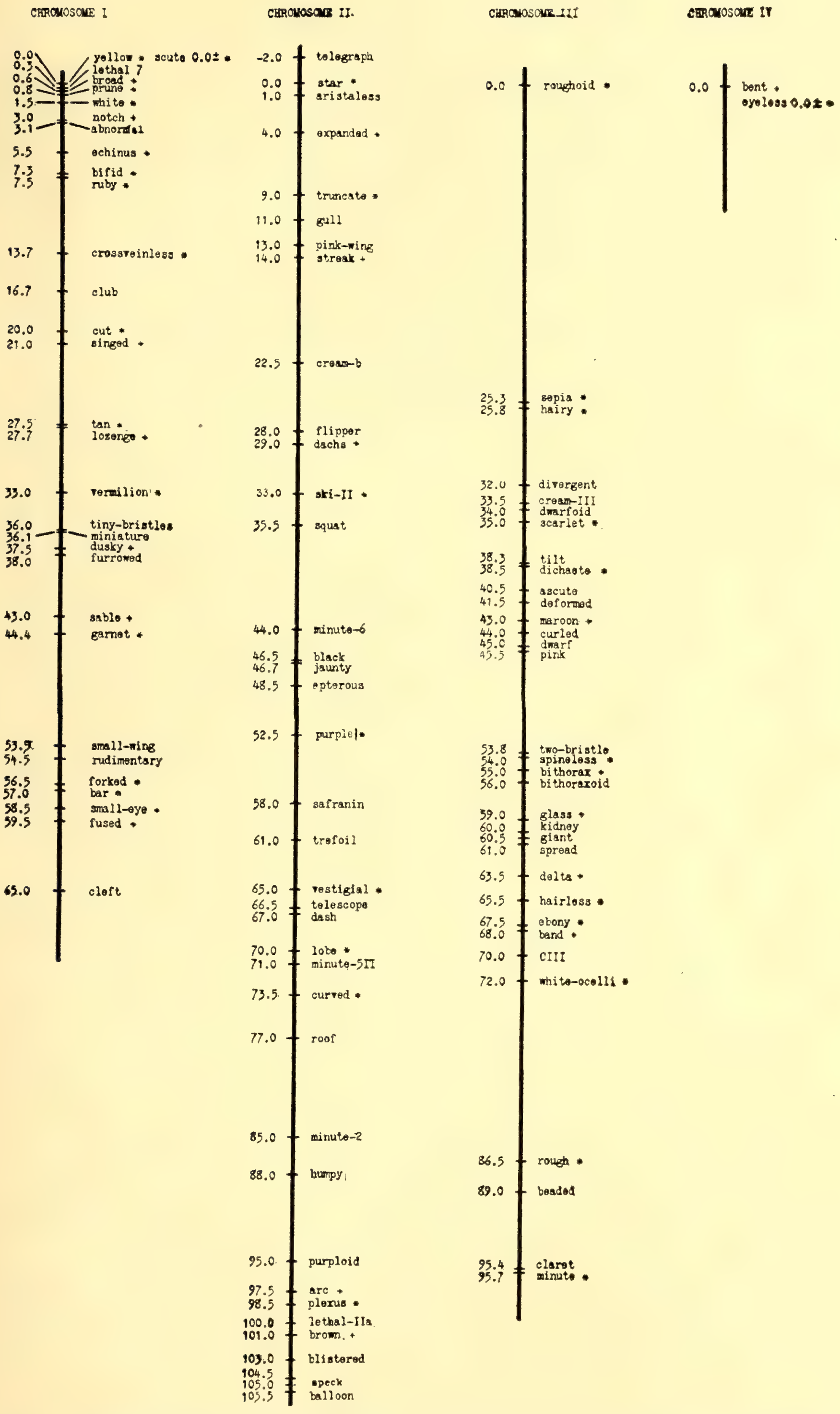
CHROMOSOME I
0.0 yellow * scute 0.0± *
0.3 lethal 7
0.6 broad +
0.8 prune +
1.5 white *
3.0 notch +
3.1 abnormal
5.5 echinus +
7.3 bifid +
7.5 ruby *
13.7 crossveinless *
16.7 club
20.0 cut *
21.0 singed +
27.5 tan *
27.7 lozenge +
33.0 vermilion *
36.0 tiny-bristles
36.1 miniature
37.5 dusky +
38.0 furrowed
43.0 sable +
44.4 garnet +
53.9 small-wing
54.5 rudimentary
56.5 forked *
57.0 bar *
58.5 small-eye +
59.5 fused +
65.0 cleft
CHROMOSOME II.
-2.0 telegraph
0.0 star *
1.0 aristaless
4.0 expanded +
9.0 truncate *
11.0 gull
13.0 pink-wing
14.0 streak +
22.5 cream-b
28.0 flipper
29.0 dachs +
33.0 ski-II +
35.5 squat
44.0 minute-6
46.5 black
46.7 jaunty
48.5 apterous
52.5 purple *
58.0 safranin
61.0 trefoil
65.0 vestigial *
66.5 telescope
67.0 dash
70.0 lobe *
71.0 minute-5II
73.5 curved *
77.0 roof
85.0 minute-2
88.0 humpy
95.0 purploid
97.5 arc +
98.5 plexus *
100.0 lethal-IIa
101.0 brown +
103.0 blistered
104.5 speck
105.0 balloon
105.5
CHROMOSOME III
0.0 roughoid *
25.3 sepia *
25.8 hairy *
32.0 divergent
33.5 cream-III
34.0 dwarfoid
35.0 scarlet *
38.3 tilt
38.5 dichaete *
40.5 ascute
41.5 deformed
43.0 maroon +
44.0 curled
45.0 dwarf
45.5 pink
53.8 two-bristle
54.0 spineless *
55.0 bithorax +
56.0 bithoraxoid
59.0 glass +
60.0 kidney
60.5 giant
61.0 spread
63.5 delta +
65.5 hairless *
67.5 ebony *
68.0 band +
70.0 CIII
72.0 white-ocelli *
86.5 rough *
89.0 beaded
95.4 claret
95.7 minute *
CHROMOSOME IV
0.0 bent +
eyeless 0.0± *

FIG. 1. Maps showing the distribution of mutant loci in the four chromosomes of *Drosophila melanogaster* (Oct. 15, 1920).

[A † sign should be added to miniature (I,36.1) and gull (II,11.0), and a * sign to black (II,46.5), speck (II,105.0), pink (III,45.5), and claret (III,95.4).]

regions of the body are located very close to one another (e. g., yellow and scute, sable and garnet, forked and bar, sepia and hairy, hairless and ebony, claret and minute). Of these neighboring mutants that one is used which interferes least with the other characters in the particular experiment, and the results of all such experiments are correlated by a simple correction made possible by knowledge of the distance between the loci. Such groups of loci, the members of which can be used as alternates, are treated as one base in the construction of maps and in linkage work.

The mutants followed by a plus sign (+) are somewhat less valuable in one or another respect, perhaps only in that of position.

Of the remaining mutants, not marked, some are exceedingly valuable in special work. As an example of these may be mentioned CIII, which prevents practically all crossing over within the right half of the third chromosome and thus is much used in holding other mutants in their proper relations in balanced stocks and while making up multiple stocks. But most of the unmarked mutants are those more difficult to separate, of poor viability, or less accurately located. Thus the mutant bithorax, which is located very close to spineless, has second rank and would be used as an alternate to spineless if the relation between the loci were accurately known. But the loci of these two mutants are so close together that as yet no double recessive has been obtained, and hence no back-cross test of the order and distance between these two loci has been made. Dominants, and recessives close to dominants, are easily and accurately placed, but there may be an error of five or even more units in the indicated positions of certain (unmarked or omitted) second-chromosome recessives whose positions are as yet known only through reference to a distant base. These are relatively unimportant mutants, and since they are not very useful, there is little incentive to determine their locations more accurately. The serial order of the loci marked with an asterisk is certain, and in only a few cases are the positions of the second-rank mutants likely to be shifted with the accumulation of data. Most of the unmarked mutants have their proper position with respect to the first and even second rank mutants, but their order with respect to each other is often uncertain in the absence of direct tests.

About 25 of the mapped loci represent more than one mutant allelomorph; thus, there are twelve allelomorphs at the white locus, and three to six allelomorphs at each of several other loci. Many of the mutants are recurrent; e. g., vermilion, notch, and rudimentary have each arisen independently on more than ten occasions.

It will be noticed that the mutant loci are not distributed at random along the chromosome maps, but are more closely spaced at the left end of the first, at both ends of the second, and throughout the mid-region of the third. This massing may correspond to a like massing of the genetic materials. When orthopteran chromosomes are in the extended phase preceding their condensation as tetrads they may be seen to consist of granules of unequal size distributed at unequal distances along the linin thread, both granule size and spacing being characteristic and relatively constant for a given chromosome. Another explanation of the uneven spacing seems more probable: The amount of crossing over between given loci, e. g., black and purple in the second chromosome, is known to vary in a definite manner with the age of the mother, with temperature, and with genetic modifiers of crossing over. Since a unit of map-distance is thus known to represent different lengths of chromosome thread, the regions in which loci appear relatively closely spaced may be regions in which one unit of map distance represents a longer section of chromosome than in those regions that appear relatively unoccupied. It may be supposed that each region of chromosome has its characteristic relation between unit of map-distance and corresponding length of chromosome. It is probably significant that in the second and third chromosomes the distribution of mutant loci is approximately symmetrical about their mid-points, for these chromosomes are V-shaped with median attachment of the spindle fibre, while the first chromosome, which has the massing at one end, has terminal attachment. The variations, from region to region, in the proportion of double crossing over, referred to above, show a distribution that is symmetrical in the second and third chromosomes. From this and other evidence it is probable that there is a definite relation between the region of chromosome, considered morphologically, and the variations in the ratio of map-unit to length of chromosome thread. When the different chromosomes are compared the average ratios are found to be practically the same for all. Thus, the lengths of chromosomes as measured directly in metaphase plates are in the ratio of 100 : 159 : 159: 12, while the known lengths of the chromosomes as mapped are in the ratio of 100 : 165 : 155 : 2.[3] This is a surprisingly close agreement between the observed lengths of the condensed chromosomes and their lengths in terms of map-units.

[1] Contribution from the Carnegie Institution of Washington.

[2] The maps in the frontispiece of the Mechanism of Mendelian Heredity were made in 1914, and are both incomplete and out of proportion. A new frontispiece, practically the same as the figure accompanying this paper, has been supplied for a revision of the Mechanism and for a French edition. A similar figure is to appear in a book by Dr. Sharp of Cornell. The best of the earlier maps of the first chromosome is that given in *Carnegie Institution Publication* No. **237,** p. 22. Partial maps of the first chromosome that are more modern have been given in connection with special papers (*Genetics,* **1,** 1916 (8); *Amer. Nat.,* **1916** (421); *J. Gen. Physiol.,* **1919** (646). A map of the second chromosome, complete to 1916, is given in *Carnegie Institution Publication* No. **278,** p. 303. No satisfactory map of the third chromosome has previously been published, though partial maps based on special work have been given (*Amer. Nat.,* **1916** (217); *Genetics,* **1919** (208).

[3] Since figure 1 was drawn (Oct. 15, 1920) a mutant (M-23) has been located at 101 in chromosome III, and there has been found to be about 1% of crossing over between bent and eyeless in the fourth chromosome. (Scarlet is at 41.5, Chromosome III.)

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Volume 7 MAY 15, 1921 Number 5

THE DIFFUSION OF AIR THROUGH WATER IN THE LAPSE OF YEARS[1]

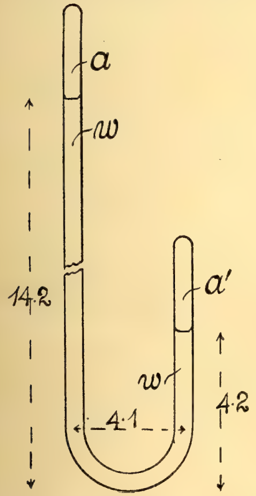
By Carl Barus

Department of Physics, Brown University

Communicated January 10, 1921

In 1912 I put up a U-tube of the form annexed, containing a charge of water, ww', below the air chambers, aa', both at nearly atmospheric pressure. The tubes were sealed by fusion cautiously, to avoid the presence of flame faces in aa'. They were then put away in a dark vault of nearly constant temperature, for short time ranges, to be examined from time to time as to the displacement of the thread of water within: for it will be seen that the meniscus under a' is at a pressure excess of $h\rho g$ as compared with the meniscus under a. If L is the total length of the thread of liquid in the tube, we may define the coefficient (κ) of diffusion (by volume) by the equation

$$v = \kappa at(h\rho g)/L$$



where v is the volume diffusing at nearly constant mean pressure in the time t through an area a, the density of liquid being $\rho = 1$. Hence,

$$\kappa = \frac{v}{at}\frac{L}{h\rho g} = \frac{1}{2}\frac{dh}{dt}\frac{L}{h\rho g},$$

where dh/dt is the loss of head per second. The factor ½ appears since the volume lost at a' appears at a, and their sum is equivalent to the loss of head. The amount of diffusion is so small that corrections may be disregarded.

The observations made before the spring of 1914 were not satisfactory because of deficiencies of method. They were, therefore, discarded. Measurement was thereafter made with the cathetometer. The following table gives the essential data for the interval of 6 years, 9 months and 6 days with the increment from leap years:

NO.	L cm.	DIAM. INSIDE cm.	AREA cm^2.	DATE	INTERVAL $t \times 10^8$ SEC.	h cm.	$10^9 \frac{dh}{dt}$	$10^{12} \kappa$
I	20.0	0.55	0.24	1914– 3–20 1920–12–26	2.14	10.12 9.89	1.07	1.10
II	15.7	0.82	0.53	1914– 3–20 1920–12–26	2.14	7.59 7.22	1.73	1.87

The two values of κ found are of the same order, but the one for tube II is definitely larger than the one for tube I. This may be due to infinitesimal difference in the separated gases (a,a') of the tubes. But since the tube II has a shorter (L) and wider column, and a meniscus nearer the bend, it is more probable that thermal correction is the cause of the difference; for the inside sectional area of II is about 2.2 times larger than that of I. The data obtained for κ with a cylindrical cartesian diver (Carnegie Publications, No. 186, Washington, 1913, p. 21) about 3 cm. in diameter of 7.05 cm^2. in area, are enormously larger than the above, reading $\alpha = 90 \times 10^{-12}$. The result is in keeping with the enormously larger area and consequent effectiveness of the solution-temperature mechanism which I have described in the preceding paper.[2] Whether in case of tube I, a condition of true diffusion has been reached will have to be ascertained in the lapse of further years. Meanwhile the question whether a long slender cartesian diver may not obviate the discrepancies hitherto encountered is worth consideration.

[1] Advance note from a Report to the Carnegie Institution, Washington, D. C.

[2] These PROCEEDINGS, **6,** 1920 (495–8).

THE INFLUENCE OF AGE ON FERTILITY IN SWINE [1]

By TAGE ELLINGER

COPENHAGEN, DENMARK

Communicated by R. Pearl, March 10, 1921

In the breeding of domestic animals the problem of fertility is of prime importance. The neglect of this essential factor has again and again caused breeders disappointments and often heavy economic losses. Thus it is desirable to study this chapter of the physiology of reproduction thoroughly and to examine the factors, internal and external, that affect the ability to produce offspring.

During an investigation on these subjects, in this laboratory the results of which are soon to be published, the writer has been confronted with the question of the influence of age of the sow on the size of her litters. Although a number of records on this and similar subjects have been published from time to time, the writer is of the opinion that the special qualities of his material, and the mathematical treatment of it that it is possible to employ, justify this further publication.

In going over the literature on the influence of age on fertility in mammals, a number of records bearing on the question can be collected. Sarah V. Jones and James E. Rouse[10] have in a recent paper give an extensive summary of the literature, to which paper I only need to refer for further information.

The records published are of two different kinds. First: results of experiments chiefly with rats (King[11]), rabbits (Hammond[8]), and guinea-pigs (Minot[14]). The experiments on swine by Mumford[15] (1917) are of further interest in this connection.

The second group consists of statistical investigations generally using as material the numerous records given in the herd books of the different breeds of domestic animals.

Carlyle and Mc Connell,[6] Pearl,[16] Bell,[3,4] Humphrey and Kleinheinz,[9] Jones and Rouse[10] give records showing that the percentage of multiple births in sheep increases with age up to five years and then again declines. Pearl[16] and Jones and Rouse[10] give similar data for cattle showing a steady increase in the percentage of twin births with age.

Swine have often been subject to investigations of this kind. One of two sources of error have always been present in these. 1. The number of individuals involved have been so limited as to make dependable results unobtainable, or 2. The method of computing has been so rough as to give room for the effect of selection. The usual way of approaching the problem has been to take from herd book records of all available litters and to refer them to the age of their mothers at their birth. In this way it is possible to get a great number of litters from young sows but rather few from older ones as the less desirable animals are disposed of early. The older sows represent only a selected group of the best of those recorded at an early age, and the average size of litters at different ages of the sows are for that reason not directly comparable. Owing to these difficulties the results are very contradictory and uncertain. Rommel[18] found a steady increase in size of litter from one to five years. Frölich and Georgs[7] report a maximum in the second litter and then a decrease in size of the the litters. Machens[12] claims a maximum at the fourth litter. Carmichael and Rice[5] give a table showing an increase in fertility to the third year followed by a decrease.

2. The material used in this investigation consists of 134 sows of native Danish breed[2] which *all* have records from all of their first ten litters. The

records are taken from the Danish herd book Vols. V and VII. All these sows have been kept in state-supervised *breeding centres* under uniform conditions and care. They have all farrowed the first time at about one year of age and from then on been bred regularly producing ca. $2^1/_4$ litter per year, practically without regard to season. The successive litters will for that reason be highly correlated with age, but it seems most convenient to use the litter-number as unit. The average of all litters, being 11.5 pigs, is far above the average shown by swine of the American type, and although these 134 sows undoubtedly are selected group, the great fertility is one of the outstanding and valuable qualities of the breed.

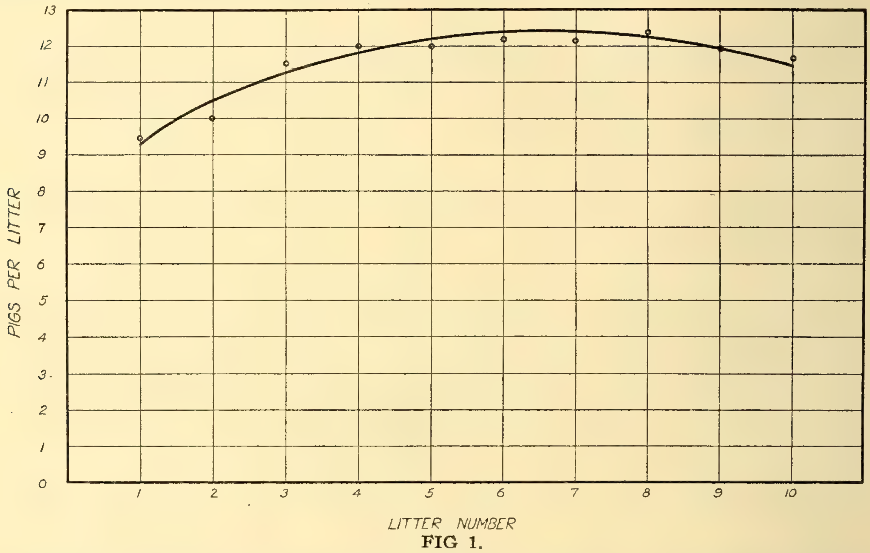


FIG 1.

The dots indicate the actual found averages for litter-size in the first ten litters. The curve is fitted to these valves.

In the second column of table 1 the average sizes of the successive ten litters are given.

TABLE I

LITTER NO.	AVERAGE SIZE OF LITTER	THEORETICAL SIZE OF LITTER
1	9.45	9.25
2	10.01	10.42
3	11.50	11.24
4	12.01	11.81
5	11.99	12.19
6	12.16	12.38
7	12.13	12.39
8	12.34	12.24
9	11.90	11.92
10	11.66	11.43
Average	11.52	11.53

Plotting the series of average values over a base indicating the successive generations (fig. 1) we see that they show an increase in fertility with age to a certain point followed by a decline. An empirical curve following these plotted points shows a type familiar to us from other biological problems. It is in fact a logarithmic curve. Pearl[17] has shown that the changes in milk flow with age in dairy cattle follow such a logarithmic curve and he has determined its equation.

The writer has now proceeded to form the equation relating fertility to to age in the material at hand, and has in the calculations used the method of moments (Miner[13]).

The general form of the equation is

$$y = a + bx + cx^2 + d \log x$$

where y denotes the size of litter and x the ordinal litter number.

In this special case the equation has been calculated to be the following

$$y = 8.414 + 0.915x - 0.078x^2 + 1.627 \log x.$$

By solving the equation for the values of x ranging from 1 to 10 the theoretical average size of the successive litters has been determined. These values are given in the third column of table 1 and they are used in plotting the curve in figure 1. This curve shows a satisfactory agreement with the empirical figures plotted out.

By putting $dy/dx = 0$ and solving for x the litter of maximum size is found to be 6.56th in order.

Summary.—1. Records from 134 sows of native Danish breed all having ten litters to their credit show an increase in fertility up to the 6.56 point in litter order followed by a decrease.

2. The curve relating fertility to age is logarithmic, its equation is calculated, and the curve fitted.

[1] Papers from the Department of Biometry and Vital Statistics, School of Hygiene and Public Health, Johns Hopkins University, No. 35.

[2] The native Danish breed is a white, fertile and hardy hog of bacon type, producing sometimes crossed with Yorkshire boars—the great proportion of the choice bacon exported from Denmark to England.

[3] Bell, A. G. 1904. The multi-nippled sheep of Beinn Bhreagh. *Science, N. S.*, **19**.

[4] Bell, A. G. 1912. Sheep breeding experiments on Beinn Bhreagh. *Ibid.*, **34**.

[5] Carmichael, W. J. and John B. Rice. 1920. Variation in farrow: With special reference to birth weight of pigs. *Univ. Illinois, Agric. Exp. Sta., Bull.* **226**.

[6] Carlyle, W. L. and T. F. Mc Connell. 1902. Some observations on sheep breeding from the Experiment Stations flock records. *Univ. Wisconsin, Agric. Exp. Sta., Bull.* **95**.

[7] Frölich und Georgs. 1911. Fruchtbarkeit und Geschlechtverhältnis beim weissen Edelschwein. *Jahrbuch für wissenschaftliche und praktische Tierzucht,* **6**.

[8] Hammond, John. 1914. On some factors controlling fertility in domestic animals. *J. Agric. Sci.,* **6**.

[9] Humphrey, G. C. and F. Kleinheinz. 1907. Observations on sheep breeding from records of the university flock. 24th *Ann. Rep. Wisconsin Agric. Exp. Sta.*

[10] Jones, S. N. H. and J. E. Rouse. 1920. The relation of dam to observed fecundity in domestic animals. I. Multiple birth in cattle and sheep. *J. Dairy Sci.,* **3**.

[11]King, H. D. 1916. The relation of age to fertility in rat. *Anat. Rec.*, **11.**

[12]Machens, A. 1915. Fruchtbarkeit und Geschlechtsverhältnisse beim vereddelten Landschwein. *Berliner Tierärtzliche Wochenschrift,* **31.**

[13]Miner, J. R. 1915. Fitting logarithmic curves by the method of moments. *J. Agric. Res.*, **3.**

[14]Minot, C. S. 1891. Senescence and rejuvenescence. First paper: On the weight of guinea pigs. *J. Physiol.*, **12.**

[15]Mumford, F. B. 1917. *The Breeding of Animals.*

[16]Pearl, R. 1913. Note regarding the relation of age to fecundity. *Science, N. S.*, **37.**

[17]Pearl, R. 1914. On the law relating milk flow to age in dairy cattle. *Proc. Soc. Exp. Biol. Med.*, **12.**

[18]Rommel, G. M. 1907. Inheritance of litter size in Poland China sows. *American Breeders Report* **3.**

VARIATION AND INHERITANCE IN SIZE IN TRYPANOSOMA LEWISI[1]

1. LIFE-CYCLE IN THE RAT AND A STUDY OF SIZE AND VARIATION IN "PURE LINE" INFECTIONS[2]

BY W. H. TALIAFERRO

DEPARTMENT OF MEDICAL ZOOLOGY OF THE SCHOOL OF HYGIENE AND PUBLIC HEALTH, JOHNS HOPKINS UNIVERSITY

Communicated by R. Pearl, May 2, 1921

The flagellate, Trypanosoma lewisi, is a non-pathogenic blood parasite occurring in various species of rats all over the world. It is known to be transmitted from rat to rat by the rat flea. This trypanosome was selected for the present work because it occurs in the latitude of Baltimore and the vertebrate and invertebrate hosts can easily be reared in the laboratory.

The general plan of the present work on size in T. lewisi is to make a careful study of size and variability in a pure line and then with this background to attempt to explain the facts observed in infections occurring in nature. After a pure line infection was obtained the following questions were attacked: (1) What are the mean and the coefficient of variation? (2) Does growing the same "pure line" in different vertebrate hosts cause significant differences in the mean size or in the coefficient of variation? (3) Does passage of the "pure line" through the invertebrate host cause significant differences in the mean or coefficient of variation? This question gives us a chance to test the possibility that passage of the "pure line" through the invertebrate host may cause a splitting up of the "pure line" into heritably diverse lines. After a study is made of these questions we can attack the final one: (4) Does an infection occurring in nature consist of a large number of "pure lines" such as has been described by Jennings[3] and others in free-living protozoa which differ among themselves but are *per se* constant in size? As will be seen later, these questions cannot be approached with any degree of exactness until a thorough study is made of the changes in mean size and variability throughout the course of an infection. The present paper deals with these changes in size and variability and with the first of the questions enumerated above. The other three questions are to be taken up in a later report.

A study of inheritance in a parasitic protozoon such as T. lewisi is of interest from several points of view. In the first place the results are of interest from a comparative standpoint when considered in the light of recent advances in our knowledge of the genetics of free-living species. In the second place, the results may be of value in the interpretation of the results of the many studies on the production of strains of parasitic organisms which exhibit new characteristics. Finally the work is the first of a program of investigations, the ultimate object of which is a study of the mechanism of the formation of new lines exhibiting such characters as arsenic-fastness and the inheritance of these characters after passage through both the vertebrate and the invertebrate hosts.

While lack of space prevents a discussion of technique in detail, it may be noted that every precaution was taken to use microscopical technique such that the trypanosomes would be free from shrinkage and distortion. All measurements were made from camera lucida outlines drawn at a magnification of ×3000. The unit used in measuring the drawings was 3 mm.; consequently all of the determinations given in this paper are in actual microns. In making the determinations 100 specimens, taken at random without selection, were drawn in each case. In isolating single organisms with which to start "pure line" infections, a sensitive mercury pipette was used in conjunction with a Barber pipette holder. Figure 1 is a diagram of a trypanosome indicating the various parts of the organism and the distances measured in this work. The names of the parts of the trypanosome run vertically and the abbreviations of the six distances run horizontally.

Size and Variation throughout a "pure line" Infection.—The infection in the rat can be divided into three periods: (1) the incubation period lasts from 1–7 days and is the time which elapses between inoculation and the first appearance of the trypanosomes in the blood. (2) The multiplication period starts with the first appearance of the trypanosomes in the blood and lasts for 10–25 days. This period is characterized by the great variations in size due to the growth and the multiplication of the trypanosomes. (3) The period of "adult" infection follows the second period and is characterized by the absence of all growth and multiplication. This period lasts from one to many weeks at the end of which time the trypanosomes disappear from the blood and the rat is immune to another infection with T. lewisi. It is apparent that in an organism which shows a cycle of growth and division such as characterizes T. lewisi, we must make all comparisons of size and variability at the same stage in the cycle, and that if there is a period in which there is no growth and division comparisons should be made during this period.

A study of the changes in the means and in the coefficients of variation of the different parts of the trypanosome throughout the course of a "pure line" infection demonstrates very clearly that there are different periods of the infection and indicates the stage at which to make comparisons of size and variability between different infections. Let us take, for example, the coefficients of variation and the means for total length in rats 116 and 105 which are shown in figure 2. The infection in rat 116 was started from a single trypanosome and rat 105 was inoculated from rat 116. In other words, although the infection in the first rat started from a single specimen and in the second rat from many specimens, all of the trypanosomes in both rats are descendants of the single organism injected into rat 116. We can make no determinations during the incubation period since no one knows where the organisms are at this time. Let us consider first the constants for rat 105. On the first day of the blood infection the mean length was 24.785 ± .423. This rose rapidly until the 5th day when it reached 30.108 ± .280. This rise continued gradually until it reached 31.412 ± .065 by the 19th day. From the 19th until the 32nd day, at which time the infection disappeared from the blood, there was no significant change in the mean. The coefficient of variation behaved in much the same manner as the mean. On the first day of the infection it was 26.52 ± 1.35%. It dropped very rapidly for the first seven days and then much slower for the next twelve days. By the 19th day it reached the low value of 3.11 ± .14% and it showed no significant change from this value throughout the remainder of the infection.

It is well to compare the conditions found in rat 116 with those in rat 105. Rat 116 is given here because a longer time elapsed before the infection reached the adult stage than was the case in any of the other infections studied. This is due probably to the fact stated above, viz., that the infection in rat 116 started from a single trypanosome while the others although they were "pure lines" were sub-inoculated from 116, and in consequence were started with a large number of specimens. As these curves are probably expressions of the resistance of the host to the parasite, we would expect that this resistance would increase more rapidly when the infection is started with a large number of trypanosomes than when it is started with only one. We cannot compare the shape of the curves in the two rats very well because there are not enough points in the curve for rat 116. One thing is probably true, however, and that is that both the mean and the coefficient of variation reach a constant value by the 25th day. At first it looks as if the mean tends to rise even until the 32nd day. That this is not the case, however, is shown by the fact that the value on the 25th day does not vary appreciably from that of the 72nd day.

The same type of result was obtained for all of the six distances shown in figure 1, namely, for the distances posterior end *to* parabasal body, parabasal body *to* nucleus, nucleus *to* anterior end, anterior end *to* end of flagellum, and width, although most of the constants do not reach as low a level as is the case with those for total length (see table 1). Curves of this nature were obtained in four rats. These results prove, what we had been led to believe from cytological evidence, viz., that there is practically no division or growth in T. lewisi after the 25th day of the blood infection.

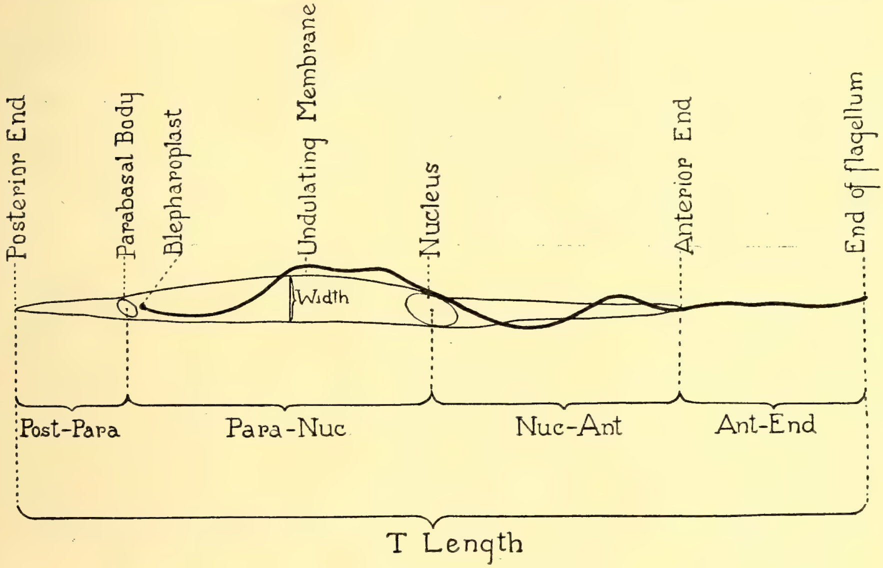


FIG. 1

Diagram of T. lewisi showing the parts of the organism and the distances measured in this work. The parts of the organism are placed vertically while the distances are placed horizontally. The latter consist of the distance from (1) posterior end to parabasal body, (2) parabasal body to nucleus, (3) nucleus to anterior end, (4) anterior end to end of flagellum, (5) total length, and (6) width.

After it was determined that there was no significant change in the mean or coefficient of variation after the 25th day we decided to make all measurements after this day. Most of the measurements which will be given in a later report were made on the 30th day of the blood infection. The fact that the trypanosomes in the blood of the rat reach what we can consider an adult stage makes size a very favorable character with which to work. It affectively eliminates growth factors from our computations.

The most interesting feature in the study of the variability exhibited by the "pure line" is that once the organism has reached the adult stage of infection the coefficients of variation are extremely low. Table 1 shows the constants for rat 105. The coefficients of variation for total length and parabasal-nucleus are 2.80% and 2.21%, respectively. It is to be noted also that these two characters show a lower coefficient of variation than any of the others. After considering this subject in great detail we have come to the conclusion that these two characters give the truest index of the variability in the "pure line." While there is not much doubt that posterior-parabasal is more variable than either of these two, we feel that much of its variability as well as that of the other distances with high coefficients is due to the difficulty of making the measurements.

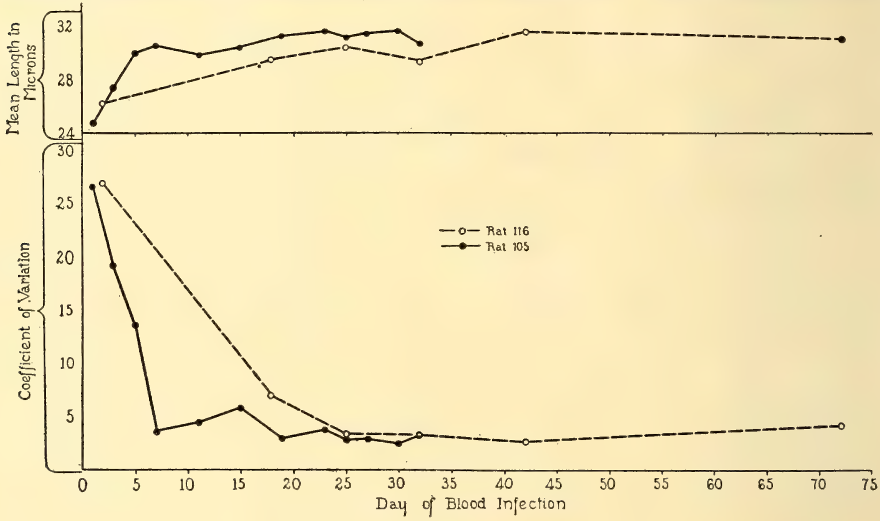


FIG. 2

Graph showing the changes in the mean and coefficient of variation for total length in rats 116 and 105. The infection in 116 was started from a single organism and 105 was inoculated from 116. The trypanosomes disappeared from the blood of rat 105 on the 32nd day and from 116 on the 72nd day.

Conclusion.—We can draw the following conclusions from this part of the work: Trypanosoma lewisi reaches an "adult" stage in its development in the rat in about 25 days after it appears in the blood. Once this stage is reached there is practically no division or growth. Due to the elimination of growth factors, the organisms show a very low coefficient of variation in "pure line" infections, provided they are measured after the "adult" stage is reached. These facts make T. lewisi a very favorable organism in which to study size and variation.

TABLE 1

CONSTANTS FOR THE SIX DISTANCES GIVEN IN FIGURE I FOR THE "PURE LINE" IN RAT 105 ON THE 25TH DAY OF BLOOD INFECTION. THE MEANS AND STANDARD DEVIATIONS ARE IN MICRONS

	MEAN	STANDARD DEVIATION	COEFFICIENT OF VARIATION
Post-Para	4.268 ± .036	.544 ± .025	12.74 ± .61
Para-Nuc	10.854 ± .016	.240 ± .011	2.21 ± .10
Nuc-Ant	9.511 ± .047	.704 ± .033	7.41 ± .35
Ant-End	6.619 ± .068	1.010 ± .048	15.27 ± .74
T. Length	31.251 ± .059	.875 ± .041	2.80 ± .13
Width	1.590 ± .015	.230 ± .010	14.47 ± .70

[1] This and a later report form a preliminary account of a series of investigations which are being carried out in this laboratory on variation and inheritance in T. lewisi.

[2] Throughout this work the term "pure line" infection has been used to designate an infection, the trypanosomes of which have all arisen from a single organism. A given "pure line" may either have been started from a single specimen or it may have been subinoculated from such an infection.

[3] See especially Jennings, H. S., *Proc. American Phil. Soc.*, **47,** 1908 (393–546). *American Nat.*, **43,** 1909 (321–337). *Ibid.*, **45,** 1911 (79–89).

MEASUREMENT OF THE DIAMETER OF ALPHA-ORIONIS BY THE INTERFEROMETER

BY A. A. MICHELSON AND F. G. PEASE

MOUNT WILSON OBSERVATORY, CARNEGIE INSTITUTION OF WASHINGTON

Communicated March 12, 1921

It was shown in these PROCEEDINGS[1] that in the application of interference methods to astronomical measures, the fringes show no decrease in visibility with the slits separated by the full aperture of the 100-inch Hooker telescope even when the seeing is poor. It was therefore decided to build an interferometer with movable outer mirrors in order to test for separations as great as 20 feet.

The interferometer bed consists of a fabricated steel beam, designed with special regard to lightness and stiffness, mounted on the end of the Cassegrain cage of the 100-inch reflector (fig. 1). Two tracks were planed on the top, true to 0.001 inch (0.025 mm.), the frame being supported on the planer as it was to be mounted on the telescope. On this beam are mounted four slides, each carrying a mirror about 6 inches (152 mm.) in diameter, inclined 45 degrees to the base. The two inner mirrors M_2, M_3, are fixed, 45 inches (114.2 cm.) apart, while those of the outer pair M_1, M_4, are movable and can be separated to a distance of 20 feet (6.1 m.). The light pencils are reflected from the outer to the inner mirrors, thence over the customary path *a, b, c, d*, in the telescope, and are viewed with an eyepiece at the Cassegrain focus *d*, where the equivalent focal length is 134 feet (40.8 m.). The fixed position of the inner mirrors gives a constant fringe spacing equal to 0.02 mm., which is easily visible with a power of 1600.

Coincidence of the two interferometer pencils at the focus is produced by adjusting the inner mirrors during the day and the outer mirrors on a star at night. Equality of path is obtained by placing the outer mirrors symmetrically on the beam, as nearly as possible, and then adjusting a double wedge of glass lying two feet within the focus in the path of one of the pencils.

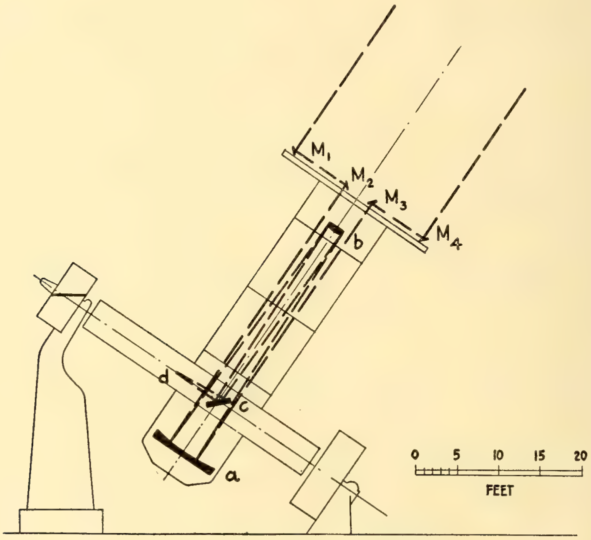


FIG. 1

Compensation for the mean thickness of the wedges is made with a plate of plane-parallel glass placed in the other pencil.

One of the wedges is fixed, the other is adjustable in a direction parallel to the inclined surfaces, a linear motion of 1 millimeter introducing an equivalent air path of about 0.09 mm. Fringes can be observed throughout a linear motion of the wedge amounting to 0.16 mm., corresponding to about 26 light waves. This range can be increased by introducing a direct-vision prism behind the eyepiece.

For comparison purposes a series of reference or "zero" fringes is obtained in the eyepiece by covering the end of the telescope tube completely save for two apertures in the beam (in addition to those of the inner mirrors) 6 inches in diameter.

When the interferometer is in perfect adjustment and the outer mirrors are close together, two separate star images are seen in the eyepiece, one formed by light reflected by the mirrors, the other by light admitted by the supplementary apertures which produce the zero fringes. Each image is surrounded by diffraction rings and crossed by the interference fringes, which appear as straight lines, alternately dark and light, perpendicular to the line joining the mirrors.

As the outer mirrors are separated, the visibility of the zero fringes remains constant, while that of the interferometer fringes gradually decreases until a point is reached where the latter vanish; as the mirrors are further separated the interference fringes should reappear, approach a secondary maximum, and again disappear, this phenomenon being repeated indefinitely.

We are interested chiefly in the first disappearance. The distance between the mirrors corresponding to extinction is the observed quantity required to determine the angular diameter of the star. Thus far it has been necessary to move the mirrors by steps, as screws are only now being installed to maintain the mirrors at equal distances.

The visibility of fringes obtained with the interferometer pointed on Vega in August, 1920, with the mirrors separated 18 feet (5.5 m.), was fully as great as with the mirrors 6 feet (1.83) apart, thus indicating that atmospheric conditions will easily permit the use of even greater separations.

Calculations of stellar diameters based on estimates of surface brightness made by Eddington, Russell, and Shapley, indicated that α Orionis, because of its relatively large diameter, would be a promising object to attempt to measure with the 20-foot interferometer. Merrill first examined the star with the apparatus used by Anderson[2] in the measurement of Capella and found a definite decrease in visibility for the maximum separation of the slits (100 inches). This was true for all position angles, thus indicating that the star was not a binary.

On December 13, 1920, the interferometer was tested by Pease on β Persei and then on γ Orionis. Both stars are known to have diameters much smaller than can be measured with this instrument. When adjusted with the mirrors separated 121 inches (229 cm.), both the zero and the interferometer fringes appeared in the eyepiece. When the instrument was directed to α Orionis the interferometer fringes could not be found. α Canis Minoris was then observed and the interferometer fringes were easily seen thus indicating that the instrument was still in adjustment. It is thus clear that the disappearance of the fringes for α Orionis was real and not due to any disturbance of the mirrors, for the seeing was good and any flexure changes involve only a very slight shift of the compensating wedge to bring the fringes into view again.

On December 14 and the nights following, the seeing was poor and as the visibility of the zero fringes was decidedly lower than on the night of the 13th, no attempt was made to work on α Orionis. With a mirror separation of 13 feet attention was directed to α Ceti, α Tauri and β Geminorum. The zero fringes were seen in every case, though much reduced in visibility, but as the interferometer fringes could not be seen at all except at intervals of better seeing, it is presumed that there was an actual decrease in visibility and that with further observation some estimate can be made of the diameter of these stars.

Assuming that the effective wave-length of α Orionis is 5.75×10^5 cm. and that the value of d is 121 inches (306.5 cm.), the angular diameter of α Orionis from the formula $\alpha = 1.22\frac{\lambda}{d}$ proves to be 0.047. An estimate of its linear diameter may be made by using a mean parallax of $0''.018$,[3] which gives a diameter of 240×10^6 miles, or slightly less than that of the diameter of the orbit of Mars.

Corrections to this value will be derived by an experimental determination of the value of λ for this particular star; by a more accurate setting of the mirrors, for the uncertainty of this measure is at least 10 per cent; and by further determination of the parallax. The angular value given above is that corresponding to a uniformly illuminated disk. A darkening toward the limb, equal to that of the sun, would require an increase in the diameter of about 17 per cent.

We wish to express our obligations to Director Hale both for his encouragement and for placing the resources of the observatory at our disposal, and to Mr. J. A. Anderson for his checking of the measures on the night of December 13.

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[1] These PROCEEDINGS, **6,** 1920 (474–475).

[2] *Mount Wilson Contributions,* No. 185; *Astroph. J. Chicago,* **51,** 1920 (263–275).

[3] The weighted mean of Adam's spectroscopic parallax, 0.012 and the trigonometric parallaxes of Elkin, 0.030, and Schlesinger, 0.016.

AN OVERLOOKED INFINITE SYSTEM OF GROUPS OF ORDER pq^2

BY G. A. MILLER

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS

Communicated April 28, 1921.

The determination of all the possible abstract groups of order pq^2 p and q being distinct prime numbers, was considered by Cole and Glover in an article published in the *American Journal of Mathematics,* vol. 15 (1893), p. 191 and by O. Hölder in a long article published in the *Mathematische Annalen,* vol. 43 (1893), p. 301. In a subsequent article, published in volume 46 of the latter journal, Hölder directed attention in foot-note on page 323 to the fact that the enumeration of these groups contained in the former of the two articles mentioned above was incomplete.

The main object of the present article is to establish the fact that there is an infinite system of abstract groups of order pq^2 which was overlooked not only by the authors already mentioned but also by others, including W. Burnside who gave an incomplete list of these groups in both editions of his well known and meritorious work entitled, "Theory of Groups of

Finite Order" 1897 and 1911. The system of groups in question consists of $q-1$ distinct groups for all values of p and q which satisfy certain conditions to be noted later while only one such group is given in the published lists. This system of groups is also interesting in view of the fact that each group of the system contains q characteristic operators including the identity.

To construct the groups of the system in question suppose that q is a divisor of $p-1$ and establish a simple isomorphism between q cyclic groups of order pq written as regular substitution groups. Let t represent the substitution of order q which permutes the corresponding letters of these q cyclic groups so that t is commutative with each of its substitutions and together with the cyclic group of order pq formed by the given isomorphism generates a regular abelian group of order pq^2. Let S_1 S_2..... S_q be substitutions of order q and of degree $pq-q$ which transform corresponding generators of the given cyclic groups of order pq into the same power belonging to exponent q modulo pq and so chosen that the product $S_1 S_2 \ldots S_q$ is commutative with t. Finally let S_0 represent a substitution of order q contained in the first one of the q given cyclic groups of order pq. The product $S_0S_1S_2 \ldots S_qt$ is a substitution of order q^2 whose q^{th} power is the substitution of order q, in the group formed by means of the said isomorphism, whose constituent is S_0. Hence it results that the pq substitutions of the group of order pq^2 thus constructed which transform into a given power belonging to exponent q modulo pq a generator of the given cyclic group of order pq can be so chosen that their q^{th} power is an arbitrary operator of order q contained in this cyclic group. The totality of these pq substitutions must correspond to itself in every automorphism of this group of order pq^2. Hence this q^{th} power must be a characteristic operator of the group.

From the preceding paragraph it results that each of the groups of order pq^2 under consideration contains q characteristic operators including the identity and that any of these operators which is of order q can be made the q^{th} power of all the pq operators of the group which transform the operators of order p in the group into a particular power. Hence there are $q-1$ distinct groups for particular values of p and q which satisfy the conditions that p and q are such primes that $p-1$ is divisible by q. These $q-1$ groups are conformal: that is, they contain the same number operators of each order. It is well known that for any prime number of q there is an infinite number of prime numbers p such that $p-1$ is divisible by q and hence there is no upper limit to the number of such distinct conformal groups.

The smallest order for which there exist at least two such conformal groups is 63. In this special case one of the characteristic operators of order 3 is the third power of the operators of order 9 which transform the operators of order 7 into their fourth powers while the other operator of order 3 is the third power of those which transform the operators of order 7 into their squares in one of the two conformal groups. In the other group the reverse is true. Hence it is not possible to establish a simple isomorphism between the operators of these two groups. In view of the elementary properties of these groups it appears strange that the wide-spread error noted above was not corrected for more than a quarter of a century, especially since the incorrect results obtained by O. Hölder have been used by various writers in extending his work.

THE GLOBE, A SIMPLE TRISOMIC MUTANT IN DATURA

By Albert F. Blakeslee

Station for Experimental Evolution, Cold Spring Harbor, N. Y.

Communicated by C. B. Davenport, March 10, 1921

In a series of articles already published,[1, 2, 3] or at the present writing in press (*American Naturalist, Genetics*), a number of recurrent mutants discovered in the Jimson Weed (*Datura Stramonium*) have been described and their peculiarities in external appearance shown to be connected with the presence of one or more extra chromosomes in their nuclei. Evidence has been presented which indicates that a given mutant of the "simple trisomic" type is conditioned by the presence of a single extra chromosome in a specific one of the 12 chromosomal sets. Such a form is called a simple trisomic mutant since in its somatic nucleus one of the 12 sets is a trisome with three homologous chromosomes instead of all the sets being disomes with two chromosomes each. The presence of an extra chromosome in a specific chromosomal set not only causes specific peculiarities in the growth and appearance of the mutant which results, but also brings about peculiarities in the inheritance of the mutant complex.

It is the purpose in the present paper to summarize the findings in regard to one of the simple trisomic mutants—the Globe—in anticipation of a more detailed paper to be published shortly in *Genetics*. The data were accumulated for the most part before the chromosomal condition in the Globe and other simple trisomic mutants had been determined by my colleague, Mr. John Belling, from studies of mitotic figures in the pollen mother cells.

The Globe was the first mutant recognized in the Jimson Weed, having been discovered in 1915. Its depressed globose capsules suggested the name. Its adult characters as well as the broad entire leaves of its seedlings render the Globe one of the easiest mutants to recognize at any stage of development. It is the only one in fact that we have been able to pick out readily in the seed pan. Since usually it has not been necessary to grow plants beyond an early seedling stage when it is desired to distinguish Globes from normals, it has been possible with this mutant to base conclusions on a larger number of individuals than could readily have been obtained if we had been dealing with the other mutant forms.

The Globe mutant differs from normals apparently in all parts of the plant. It shows a complex of characters readily recognized, whether the plants in question have purple or white flowers, many or few nodes, and spiny or smooth capsules. Globes, like other mutants of this type, are slower in growth than normals; and in competition with normals are liable to be crowded out by them.

TABLE 1

Globes Selfed and Crossed Reciprocally with Main Line Normals. Plants Pollinated in Greenhouse, 1916–17. Records Taken in Seed Pans in Greenhouse, 1917–18

ORIGIN OF SEED	SEEDS PARENTS	SEEDS PLANTED	SEEDLINGS	PER CENT SEEDLINGS	GLOBES	NORMALS	PER CENT GLOBES	RATIO G:N
Globe Selfed	27	3015	1782	59.10	400	1382	22.45±0.676	1:3.46
Globe x Normal	27	2628	1935	73.63	500	1453	25.84±0.671	1:2.87
Normal x Globe	27	2771	1812	65.39	53	1759	2.92±0.269	1:33.19

Percentage difference between offspring from Globe Selfed and from Globe x normal =3.39 ± 0.935. Diff./E. Diff. = 3.59.

Early breeding work with the Globe indicated that the manner of its inheritance was not in accord with any simple Mendelian interpretation. A more extensive series of selfs and crosses was therefore carried out in order to determine more accurately the extent to which the Globe complex could be transmitted. In table 1 is presented a summary of the experiment. In this table are given the number of seeds planted and the percentage of recordable seedlings which they produced. It is obvious that a difference exists between the inheritance through the male and that through the female parent. The egg cells of Globes, whether fertilized by Globe pollen or by pollen from normal plants, transmit the Globe complex to only about 25 per cent of the offspring. Globe pollen, when used on normal plants, transmits the character to only about 3 per cent of the offspring This figure, though small, is too large to be accounted for by the occurrence of new Globe mutations, in view of the rarity of Globes from normal parents in comparable material. The normal offspring of Globes seem to be true normals so far as their breeding behavior is concerned.

If 3 % represents the average number of Globe pollen grains which transmit the mutant character, we should expect a higher proportion of Globe offspring when Globe pollen is used than when pollen from normals is used on Globe females. The reverse, however, is actually the case, and Globes selfed produced a lower percentage of mutant offspring than Globes pollinated by normals. The explanation may possibly be connected with a difference in vitality between seedlings produced by self and those produced by cross pollination. The fact is at least suggestive that a lower per-

TABLE 2

Main Lines and Occurrence of New Globe Mutations from 1915 to 1920. The First Mutation Found in 1915 Is Not Included. Under "Type," "P" Stands for the Mendelian Character Purple and "W" for White, "ar" for Armed Capsules and "in" for *Inermis*, "M" for Many Nodes and "F" for Few Nodes

NO.	TYPE	ORIGIN	SEED PARENTS	NOR.	GL.	% GLOBES
1	Par M	Washington, D. C.	50	3313	6	0.181 ± 0.005
2	War M	Washington, D. C.	19	1292		
3	War F	Erfurt, Germany	9	854		
4	Win F	Erfurt, Germany	20	1479		
5	Par M	Staten Island, N. Y.	17	933		
6	War M	Staten Island, N. Y.	35	1462		
7	War M	Bronx, N. Y.	13	1186		
8	War M	Woodbury, N. Y.	7	377		
9	Par M	Cold Spring Harbor, N. Y.	13	844	2	0.236 ± 0.112
10	Par M	Charlotte, Va.	5	394		
11	Par M	Danville, N. C.	6	595		
12	War M	Lloyds Neck, N. Y.	1	92		
13	War F	J. L. Childs	6	625		
14	War F	Naples, Italy	6	627		
15	Par M	D. Stramoinum gigantea	5	408		
16	Par F	D. Leicharti	5	370		
17	Win F	D. Laevis	5	701		
18	Par M	D. Ferox	5	400	1	0.249 ± 0.168
19	War F	D. Quercifolia	7	653		
Totals—Main Lines Inbred			234	16605	9	0.054 ± 0.012
F_1's			159	7086	3	0.042 ± 0.016
Back crosses			57	2064		
Extractives			215	12353	12	0.091 ± 0.018
Totals—Crosses and Extractives			431	21503	15	0.070 ± 0.012
Grand totals			665	38108	24	0.063 ± 0.009

centage of recordable seedlings was obtained when Globes were selfed than when they were crossed with normals.

Simple trisomic mutants produce dimorphic gametes with 12 and 13 chromosomes instead of gametes with uniformly 12 chromosomes as is characteristic of normals. Two types of egg cells should be produced in equal numbers and, on the assumption that in general only pollen grains with 12 chromosomes are able to assist in fertilization, an equal number of Globes and normals should appear in the offspring of Globes. That mutants and normals are not thrown in equal numbers is probably due to the weaker vigor of the mutant zygotes.

From the foregoing, it is concluded (*a*) that the mutant complex is transmitted primarily through the egg cells and (*b*) is transmitted not at all or only to a slight extent through the pollen; (*c*) that the deviation in the offspring from an equality of Globes and normals is due to the lessened vitality of the mutant forms; (*d*) that the higher percentage of Globes obtained by crossing Globes with normals, over selfing, is due to an increased vigor brought about by the cross.

Simple trisomic mutants are characterized by a considerable proportion of defective pollen grains. The Globe has about 8 per cent bad grains

TABLE 3

OCCURRENCE OF NEW GLOBE MUTATIONS FROM MUTANT PARENTS

	MUTANT SELFED SEED				MUTANT X NORMAL SEED				NORMAL X MUTANT SEED			
PARENT	PAR.	NOR.	GLOBES	% GL.	PAR.	NOR.	GLOBES	% GL.	PAR.	NOR.	GLOBES	% GL.
Poinsettia	11	968	4	0.41	5	389	1	0.26	3	591	1	0.17
Poinsettia var. Wiry	2	189	0		1	15	0		1	225	1	0.44
Cocklebur	20	1859	1	0.05	8	1047	1	0.10	6	1167	0	
Ilex	8	855	0		5	538	1	0.19	4	975	0	
Mutilated	16	871	1	0.11	8	351	0		4	552	2	0.36
Sugar Loaf	8	413	1	0.24	3	367	0		1	197	0	
Rolled	8	705	1	0.14	2	222	4	1.80	1	245	1	0.41
Reduced	6	707	2	0.28	1	127	1	0.78	1	215	0	
Buckling	8	856	1	0.12	1	106	0		1	246	0	
Glossy	7	897	1	0.11	1	143	0		1	253	0	
Microcarpic	18	994	1	0.10	5	178	0		4	353	0	
Spinach	13	625	8	1.26	3	187	4	2.09	2	377	0	
Totals	125	9939	21	0.211 ±.031	43	3670	12	0.326 ±.064	29	5396	5	0.093 ±.028

Percentage difference between offspring from mutant selfed and mutant x normal =0.115 ± 0.071. Diff./E. Diff. = 1.62.

Percentage difference between offspring from mutant x normal and normal x mutant =.233 ± .070. Diff./E. Diff. = 3.33.

which is the lowest average for any of the Datura mutants of this type yet investigated. Normal plants show only about 2.7 % of defective grains.

Before the cause of the Globe mutation was discovered, an attempt was made to increase by selection the proportion of Globes in the offspring of selfed Globe parents. Globes were accordingly inbred for ten generations but the proportion of mutants and normals in the offspring was not noticeably affected.

Inasmuch as the Globe is one of the most readily recognized of all the Datura mutants and has been sought for in our cultures since its first appearance in 1915, it is possible to give more reliable figures in regard to the frequency of its new appearance than could be done for the mutants discovered later. In table 2 is summarized the occurrence of Globe plants in offspring from selfed parents of the 19 different main lines. The various lines have the color, spine and node characters indicated in the table. It is seen that new Globe mutations have occurred in all the lines considered together in about 0.05% of the offspring which is about one out of 2000 individuals. The offspring from crosses between main lines have shown new Globe mutations in about the same proportion as in the main lines themselves.

We have just discussed the occurrence of Globe mutations from normal parents. It will be of interest to consider the occurrence of this mutant from parents belonging to the other mutant forms. In table 3 are shown the percentages of new Globes that have occurred from selfing the various mutants and from crossing them reciprocally with normals. The percentage of Globes appears to be somewhat higher than from the main line parents. It is of interest to note that when mutants are used as males in crosses with normals, the percentage of Globe mutations is reduced to about the average expected from the assemblage of main line parents. A comparison of table 3 with table 1, however, will be instructive and lead us to the conclusion that the egg cell is not only more effective in transmitting the mutant character, once it has arisen, but is also more effective in originating these mutations than is the pollen grain.

[1] Blakeslee, A. F. and B. T. Avery, *J. Heredity*, **8**, 1917 (125–131).

[2] Blakeslee, A. F. and B. T. Avery, *Ibid.*, **10**, 1919 (111–120).

[3] Blakeslee, A. F., John Belling and M. E. Farnham, *Science, N. S.*, **52**, 1920 (388–390).

STUDIES OF MAGNITUDES IN STAR CLUSTERS
XIII. VARIABLE STARS IN N. G. C. 7006

By Harlow Shapley and Beatrice W. Mayberry

Mount Wilson Observatory, Carnegie Institution of Washington

Communicated by G. E. Hale, March 22, 1921

The faintest and most distant variable stars on record have been found on photographs of the globular cluster N. G. C. 7006 made with the 60-inch and 100-inch reflectors at Mount Wilson. The new variables are of considerable value in measuring the size of the galactic system, because they are members of one of the remotest stellar groups now known and serve to determine its distance and dimensions.

The position of the cluster for 1900.0 is:

$$\text{R.A.} = 20^h\, 56^m\!.8, \quad \text{Decl.} = +15^\circ\, 48'$$

Its galactic coördinates are:

$$\beta = -20^\circ, \quad \lambda = 32^\circ$$

and its apparent diameter on the Franklin-Adams charts is three-fourths of a minute of arc.

By the usual photometric methods for determining the distances of globular clusters, the parallax of N. G. C. 7006 has been estimated to be:

$\pi = 0''.000015$, from the parallax-diameter correlation.[1]
$\pi = 0.000014$, from photographic magnitude of brightest twenty-five stars.[2]
$\pi = 0.000016$, from photovisual magnitude of brightest thirty-eight stars.[3]
$\pi = 0.000016$, from integrated visual magnitude.[4]

The foregoing accordant results make it fairly certain that N. G. C. 7006 is somewhat more distant than any other globular cluster in the present lists;

but it is now possible with the aid of the variable stars to determine the distance much more accurately.

The following photographs of the cluster have been examined in the stereocomparator for the detection of variable stars.

TABLE 1

PLATE No.	PLATE Kind	DATE	TELESCOPE	EXPOSURE TIME
4168	S27	1917 Nov. 10.387	60-inch	15^m
5168a*	S30	1919 Dec. 24.284	60-inch	30
5402	S30	1920 July 16.622	60-inch	70
5500	S30	1920 Aug. 10.515	60-inch	50
P 9	S30	1920 Sept. 11.428	100-inch	30
P 23	S30	1920 Sept. 12.381	100-inch	40
P 29	S30	1920 Sept. 12.556	100-inch	30
5601	S30	1920 Sept. 14.312	60-inch	45
P 50	S30	1920 Sept. 15.379	60-inch	60

*Plate 5168a was made by Mr. Hoge, the others by Mr. Shapley.

TABLE 2

STAR	X	Y
1	—177″.6	+113″.8
2	—38.0	—38.0
3	—27.6	+32.8
4	—24.2	—42.2
5	—24.2	+36.2
6	—15.5	—44.8
7	0.0	—38.0
8	+31.0	+13.8
9	+36.2	+15.5
10	+38.0	—13.8
11	+141.4	+48.3

Seven of the eleven variables found on the above plates show conspicuous variation in the course of one night, and the other four appear also to be short period variables. In magnitude and range they are very much alike, and in this cluster, as in Messier 3, ω Centauri and other systems, the variables are between one and two magnitudes fainter than the brightest stars. It is probably safe to assume, therefore, that these faint variable stars are typical short-period Cepheids, similar to those commonly found in globular clusters.

The coördinates of the variable stars, referred to the center of the cluster, are given in seconds of arc in table II. Nos. 2 and 5 were found or suspected by Miss Ritchie, the others by Miss Mayberry.

From fifty-three measures of the brightness of the eleven variable stars on the plates made with the 60-inch reflector, a median magnitude of 18.96 is found. The median for the seven that appear most certainly to be short period Cepheids is also 18.96. The photographic magnitudes[4] of the variables are based on the results for the bright stars of the cluster, published in *Mt. Wilson Contr.*, No. 156, p. 4, 1918.

The observed range in photographic magnitude for the eleven variables is 0.83; for the seven variables it is 0.99.

If we adopt from *Mt. Wilson Contr.*, No. 151, the value —0.23 as the median photographic absolute magnitude of short period Cepheids, we have $m - M = 19.2 = 5\ (\log d - 1)$, from which the distance, d, is found to be 69,000 parsecs. Expressed as a parallax, this is $0''.0000145$—a highly satisfactory confirmation of the former results.

Although N. G. C. 7006 is the smallest known cluster in angular diameter, its linear dimensions are comparable with those of Messier 13. The diameter probably exceeds 300 light years, since two of the variables are more than 170 light years distant from the center of the cluster.

It is of interest to recall that the study of the brightest stars in N. G. C. 7006 has shown no effect of selective scattering of light in space. It also appears, from the agreement of the parallax based on angular diameter with the values based on magnitude methods, that the general non-selective absorption of light is inappreciable in the direction of N. G. C. 7006, even when the distance involved is, as in this case, 225,000 light-years.

[1] *Mt. Wilson Contr.*, No. 152. p. 13, 1917; *Astroph. J. Chic.*, **48,** 1918 (166).

[2] *Mt. Wilson Contr.*, No. 152, p. 14, *n*-1, 1917; *Ibid.*, **48,** 1918 (167).

[3] *Mt. Wilson Contr.*, No. 156, pp. 1–6, 1918; *Ibid.*, **49,** 1919 (249–54).

[4] *Mt. Wilson Contr.*, No. 161, pp. 16–18, 1918; *Ibid.*, **50,** 1919 (122–4).

THE SKIN TEMPERATURE OF PACHYDERMS

BY FRANCIS G. BENEDICT, EDWARD L. FOX, AND MARION L. BAKER

NUTRITION LABORATORY, CARNEGIE INSTITUTION OF WASHINGTON, BOSTON, MASS., AND NEW YORK ZOÖLOGICAL PARK, NEW YORK CITY

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A knowledge of the temperature of the skin is of great assistance in interpreting the laws governing the loss of heat from the animal organism, but with warm-blooded animals, such as birds and many mammals, the skin temperature is determined only with difficulty, because of the covering of hair or fur. With humans, also, the protective clothing interferes with the accurate determination of skin temperature, for numerous observations have shown that even under ordinary clothing the skin temperature of man is by no means constant and, due to the protective covering, is undoubtedly much higher than it would otherwise be. It seemed, therefore, that records of the skin temperature of certain hairless animals would be of value, since the factor of covering would thus be eliminated. Accordingly, we took advantage of the excellent material at the New York Zoological Park and of the friendly coöperation of Mr. Raymond L. Ditmars to make such observations, and as a result are able to report the surface temperatures of two elephants, one rhinoceros, and a hippopotamus, all presenting large hairless surface areas.

The method employed was that described in an earlier communication.[1] The apparatus consists of two copper-constantan thermo-junctions, with special means of application to the skin to secure the true skin temperature rather than the resultant of skin and environmental temperature. One junction is kept in a thermos bottle with a carefully calibrated thermometer, which is read practically every minute or two, and the other junction applied to the skin. The difference in temperature between the two junctions is noted on a galvanometer, whose "constant" is likewise checked every few minutes.

The natural inquisitiveness and restlessness of the elephants made the task of recording the skin temperature somewhat trying, although in no sense hazardous. The rhinoceros presented an unexpectedly favorable opportunity for measurement, while the uncertain temperament of the hippopotamus interfered greatly with the securing of records of his temperature. An especially advantageous feature of this series of observations is the fact that the temperature of the environmental air in the elephant house at the New York Zoölogical Park remains relatively constant at about 19.5° Centigrade, night and day.

Excluding records obtained at partially protected points, such as the back of the ear, the groin, and axilla, we found with a female Indian elephant a range in skin temperature from a minimum of 20.8° C. at a position on the forehead, midway between the two eyes, to a maximum of 29.6° C. on the shoulder. With a male African elephant the range was from 21.4° to 28.3° C. The relation between surface temperature and the distribution of the blood was strikingly evident in the temperature records on the ear, on the back side of which the veins are very readily observable. Near the veins the skin temperature was high. Owing to the cracks and crevices in the thick elephant skin, difficulty was at times encountered in securing records of surface temperature, especially on the sides of the body, since the tendency was for the warm air to rise, with a distinct circulation of air through the cracks in the skin. Every attempt was made to use as smooth a piece of skin as possible for the temperature measurements. The elephant, therefore, exposed to an environmental temperature of approximately 19.5° C., had a skin temperature which averaged not far from 25.5° C. While there were rather large differences at times between the two elephants, the general picture showed a reasonably uniform bilateral temperature distribution.

The rectal temperature of elephants has not been frequently reported. Dr. W. R. Blair of the New York Zoölogical Society found in his measurements that the rectal temperature of the elephant ranged from 97.2° F. (36.2° C.) to 98° F. (36.7° C.). Using a clinical thermometer, we secured the rectal temperature of the male elephant and found it to be 35.9° C. On two occasions the thermometer was thrust quickly into freshly passed feces, and temperature records secured of 36.2° and 36.7° C., respectively, the large mass of feces losing heat slowly. Thus the elephant apparently has a rectal temperature slightly lower than the average commonly assumed for man.

The skin of the rhinoceros is much smoother than that of the elephant, making possible much more satisfactory measurements. This animal was studied almost exclusively on one side of the body and gave temperature measurements ranging from 24.1° to 27.9° C. In general the temperatures were somewhat warmer in the lower part of the body. An approximate average of 26.2° C. can be taken as the general skin temperature of the rhinoceros (living in an environmental temperature of 19.5° C.), a value but slightly higher than that found with the two elephants. According to Dr. Blair the rectal temperature of the rhinoceros ranges from 99.7° F. (37.6° C.) to 100° F. (37.8° C.). Our single observation of the rectal temperature of this animal showed a value of 37.4° C.

Although considerable difficulty was experienced in securing the skin temperature of the hippopotamus, sufficient measurements were obtained to give a fairly clear picture of the probable temperature distribution. The temperatures ranged from 20.8° C. on the back to as high as 30.9° C. on the belly. An average figure for the temperature of the whole skin is probably not far from 25° C. In general the lower part of the body was found to be much warmer than the back, the temperature differences being greater with the hippopotamus than with the other three animals. This may be partly due to the fact that there was considerable vaporization of water from the skin, for the skin of the hippopotamus was much moister than that of the other two animals, although he had been out of water for several hours in an environmental temperature of approximately 19.5° C. prior to the actual temperature measurements. As a matter of fact, the temperatures of the dry and wet bulbs near the animal were 19.6° and 11.9° C., respectively. We have been unable to find any records of the rectal temperature of the hippopotamus.

To sum up, then, it can be seen that this group of animals, with a rectal temperature essentially that of man, lives in captivity in a temperature environment of about 19.5° C. with a continuous surface temperature, on the average, but 6 to 7 degrees above the environmental temperature. Man, independent of external temperature as a result of the use of clothing, has adjusted himself to a very much higher skin temperature, although this is still a greatly variable factor, averaging under clothing not far from 33° C. The direct measurement of the heat loss of these huge animals presents engineering problems rather difficult to solve. It is to be hoped that the indirect method of calorimetry (through the measurement of the oxygen consumption and the carbon-dioxide production) under these conditions of internal, environmental, and skin temperatures, may be employed to contribute further to our knowledge of the fundamental laws governing heat loss in the warm-blooded animal unprotected by a fur coating. It must be recognized, however, that with the elephant, rhinoceros, and hippopotamus, as the name of this group of animals implies, extraordinarily thick skins have been developed and thus in a way these animals carry around, instead of fur, an extra heavy plating of epidermis which, without doubt, measurably retards heat loss.

The details of this study are to be found published in the *American Journal of Physiology* for July, 1921.

[1] Benedict, Miles, and Johnson, these PROCEEDINGS, **5**, 1919 (218).

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PROCEEDINGS
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NATIONAL ACADEMY OF SCIENCES

Volume 7 JUNE 15, 1921 Number 6

FEEDING EXPERIMENTS WITH MIXTURES OF FOODSTUFFS IN UNUSUAL PROPORTIONS*

By Thomas B. Osborne and Lafayette B. Mendel

Connecticut Agricultural Experiment Station, and the Sheffield Laboratory of Physiological Chemistry in Yale University

Read before the National Academy of Sciences, April 26, 1921

Ever since Magendie[1] announced, in 1816, that the protein group of foodstuffs represents an indispensable component of the dietary of the higher animals, students of nutrition have been interested in the part played by the other nutrients that commonly enter into our food intake. Are fats or carbohydrates or both equally essential for successful nutrition? We pointed out some time ago that the reason why there is no available information respecting the actual requirement of the healthy mammal for fat is attributable to the experimental difficulties heretofore inherent in its solution.[2] Until the significance of the accessory food factors now known as vitamines was appreciated studies of the physiological value of mixtures of different foodstuffs, etc., were liable to lead to failure and erroneous conclusions, not because the supply of energy or protein or salts was inadequate but because other unrecognized and unidentified essentials were lacking.

Since it has become possible, by taking account of these various newly appreciated properties of foods, to devise rations in which essentially one factor at a time may be altered, the investigation of the rôle of the individual nutrients has become more promising. Accordingly we have already succeeded in raising young animals (albino rats) from an early age to adult size on diets which were exceptionally poor in fats.

The food mixtures consisted of the residues from extracted lean meat as a source of protein, starch, inorganic salts, together with small quantities of alfalfa and dried brewery yeast furnishing the vitamins A and B. Analyses of the rations consumed indicated that the maximum intake of fat at any time did not exceed 0.3 per cent of the food eaten. Inasmuch as all the animals starting on the diet with a body weight of approxi-

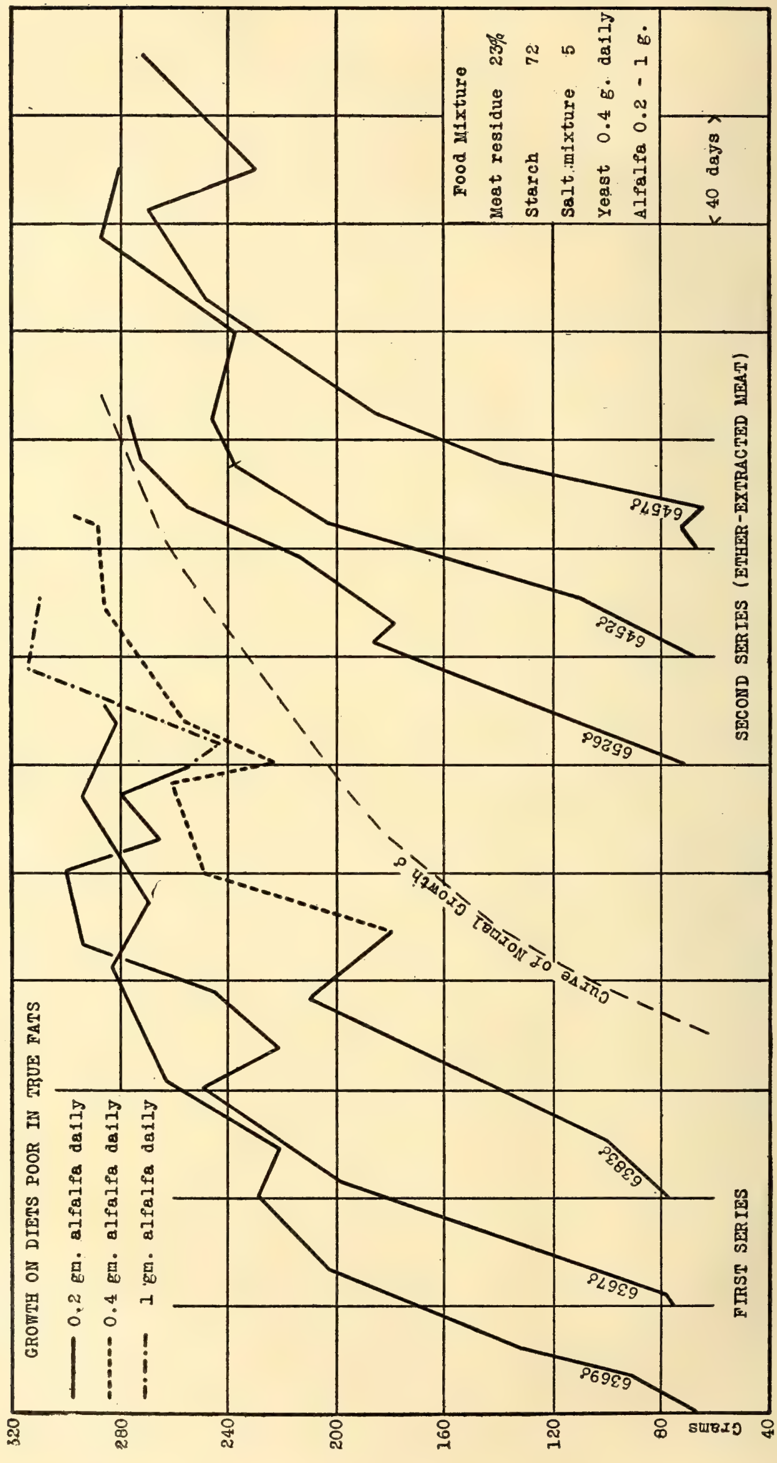
GROWTH ON DIETS POOR IN TRUE FATS
0.2 gm. alfalfa daily
0.4 gm. alfalfa daily
1 gm. alfalfa daily
Food Mixture
Meat residue 23%
Starch 72
Salt mixture 5
Yeast 0.4 g. daily
Alfalfa 0.2 - 1 g.
< 40 days >
Curve of Normal Growth ♂
6369♂
6367♂
6383♂
6526♂
6452♂
6457♂
FIRST SERIES
SECOND SERIES (ETHER-EXTRACTED MEAT)
320
280
240
200
160
120
80
40
Grams

CHART 1

mately 70 gm. have quadrupled their weight within the usual time, and appear as well nourished as companion rats on diets containing liberal portions of butter fat or lard, we cannot avoid the conclusion that if true fats are essential for nutrition during growth the minimum necessary must be exceedingly small. Drummond[3] has also secured some success on diets practically devoid of fat with rats receiving an approximate daily intake of neutral fat amounting to 14 mg. He too concludes that unless this minute amount of fat plays as important a rôle in the metabolism of the organism as do the minute quantities of such substances as the accessory factors, it is reasonable to suggest that pure fats are dispensable constituents of the mammalian diet.

The outcome of all these investigations leads one to question seriously the contentions made, particularly during the recent war, that fats as such play some unique rôle in maintaining well-being; and further, as Maignon[4] supposes, that they play an important rôle in the utilization of protein,—a rôle which carbohydrates are powerless to fill. On the other hand the demonstrations afforded by our experiments should not be construed to minimize the great value of fats as a source of energy in the usual dietary, as well as their peculiar advantages in culinary procedures.

As we have recently pointed out[5] carbohydrates are ordinarily regarded as indispensable components of the food intake. This belief is based on the presence of more or less carbohydrate in the food mixtures consumed by man and the higher animals, and the fact that sugar is a constant constituent of the blood. It is almost universally taught that carbohydrate is essential for the proper metabolism of fats in addition to any other functions that it may perform in the body; for ketone substances may be excreted in diabetes when sugar fails to be burned up in the normal manner in the organism. On the other hand it is assumed that glucose can be formed from the protein molecule or its amino-acids under certain conditions in the metabolism so that one could conceive carbohydrate to become available for the special needs of fat metabolism and other purposes without being specifically furnished as *preformed* carbohydrate in the diet. The current opinion, summarized by one recent writer,[6] maintains that "carbohydrates are the most economical of the foodstuffs, both physiologically and financially. They are the greatest sparers of protein. Ingestion of fat has for its object the relieving of the intestine from excessive carbohydrate digestion and absorption. Ingestion of fat in too large quantities leads to digestive disturbances, and if carbohydrates are entirely abandoned, to acetonuria."

Not long ago we announced that rats receiving a diet in which the amount of digestible carbohydrate was at most exceedingly small can grow from an early age to adult size. The rations which we fed included pro-

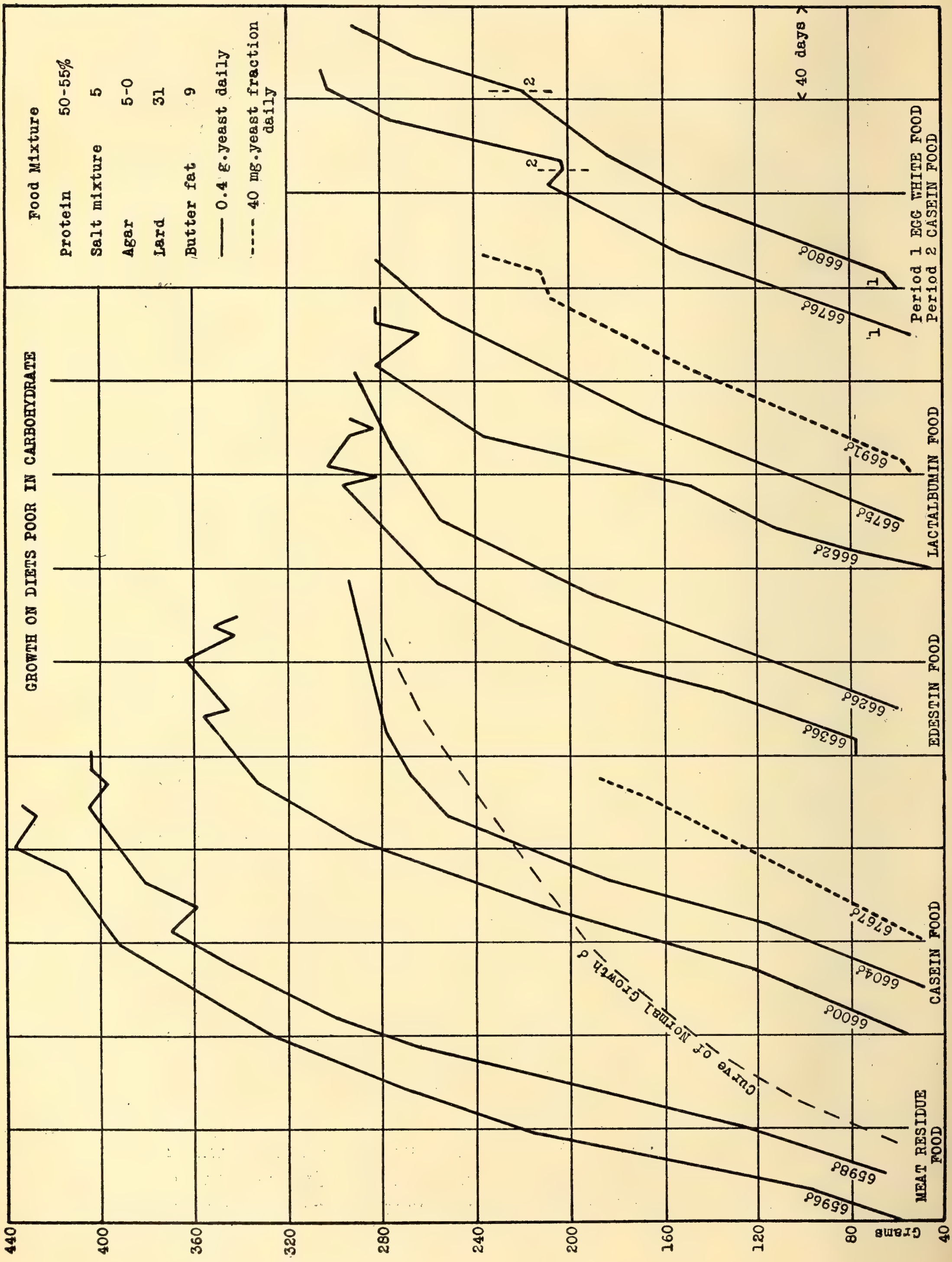
GROWTH ON DIETS POOR IN CARBOHYDRATE
Food Mixture
Protein 50-55%
Salt mixture 5
Agar 5-0
Lard 31
Butter fat 9
0.4 g. yeast daily
40 mg. yeast fraction daily
< 40 days >
2
2
1
1
6680♂
6676♂
Period 1 EGG WHITE FOOD
Period 2 CASEIN FOOD
6691♂
6675♂
6662♂
LACTALBUMIN FOOD
6626♂
6636♂
EDESTIN FOOD
6767♂
6604♂
6600♂
CASEIN FOOD
Curve of Normal Growth ♂
6598♂
6596♂
MEAT RESIDUE FOOD
440
400
360
320
280
240
200
160
120
80
Grams
40

CHART 2

tein—casein, edestin, or lean beef which had been thoroughly extracted with boiling water—inorganic salts, agar-agar, lard, butter fat and 0.4 gm. daily of dried brewery yeast furnishing vitamin B. This amount of yeast can scarcely be regarded as a significant source of available carbohydrate. Success was likewise attained in experiments in which no agar-agar was introduced. In the latter case the only obvious sources of preformed carbohydrate aside from the yeast employed were such carbohydrate impurities as might still adhere to the protein preparation fed.

The newest experiments conducted with materials from which the carbohydrate was even more rigorously excluded have not yet been brought to a conclusion; but in these also considerable growth at a normal rate has already been attained. It will be observed that the dietary conditions under which this series of nutrition experiments has been conducted are such as are reputed to lead to ketosis and ketonuria in human subjects.

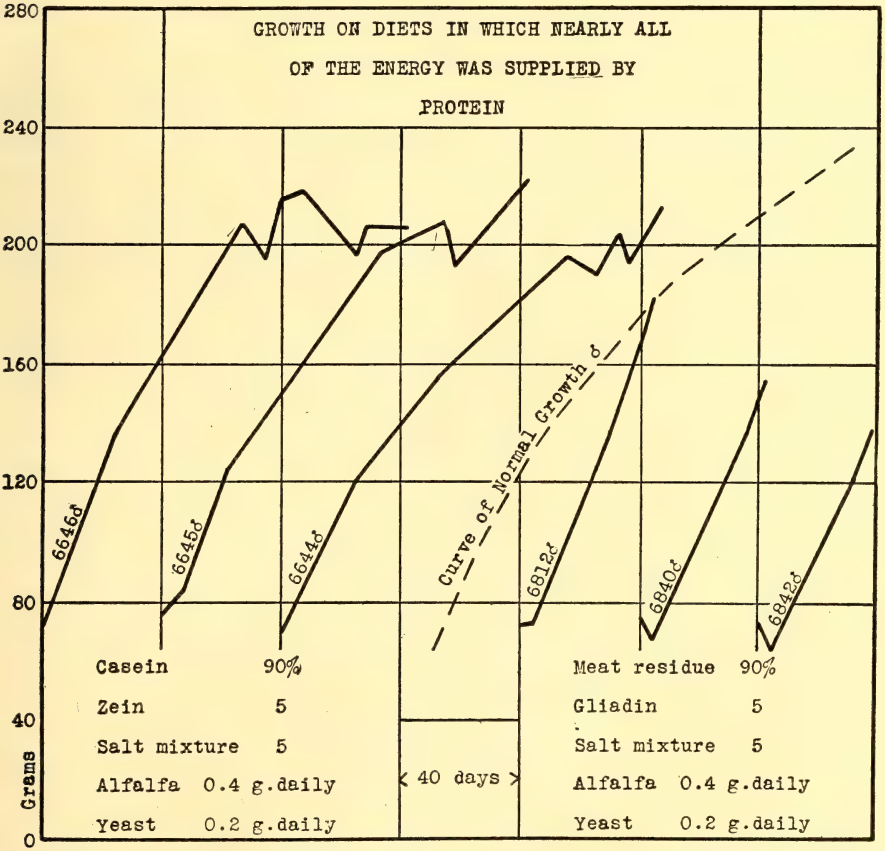

CHART 3

In view of the success of the foregoing experiments we have investigated the possibility of nourishing animals on food mixtures from which both carbohydrates and fats have been eliminated as far as possible. The attempts have thus far resolved themselves essentially into feeding a diet of which more than 90 per cent consists of protein, along with 5 per cent of inorganic salts and vitamins in the form of small daily doses of dried alfalfa and brewery yeast.

Although it has been demonstrated by experiments of Pflüger[7] that a carnivorous animal can be kept alive and maintained in activity of considerable periods on an exclusive diet of meat, it has been stated that omnivora and herbivora cannot survive on such a ration. In our own trials[8] in which the vitamin-bearing substances, representing 4–8 per cent of the food eaten, were the only noteworthy sources of either fat or carbohydrate, a number of animals have already approximately tripled their size, growing at essentially a normal rate for their species.

In all of the experiments on the unusual diets reported in this communication the protein content of the food mixtures was unusually high. Accordingly the question may well be raised whether a comparatively high protein content in the dietary during growth is not actually advantageous rather than detrimental as has been suggested by certain writers. Whether animals will attain full adult size and normal function on diets furnishing protein as the almost exclusive source of energy and tissue substance cannot be foretold at the present time. In any event the experience gained from these newest studies raises a number of important problems of physiological interest and also suggests new possibilities of investigation from novel standpoints.

* The expenses of this investigation were shared by the Connecticut Agricultural Experiment Station and the Carnegie Institution of Washington, D. C.

[1] Magendie, F., *Ann. Chim. Phys.*, **3**, 1816 (66).

[2] Osborne, T. B., and Mendel, L. B., *J. Biol. Chem.*, **45**, 1920 (145).

[3] Drummond, J. C., *Proc. Physiol. Soc., J. Physiol.*, **54**, 1920 (xxx).

[4] Maignon, F., *Récherches sur le rôle des graisses dans l'utilisation des albumnoides.* Lyon, 1919.

[5] Osborne, T. B., and Mendel, L. B., *Proc. Soc. Exper. Biol. Med.*, **18**, 1921 (136).

[6] Lusk, G., *The Science of Nutrition.* Philadelphia, 1917, p. 288.

[7] Pflüger, E., *Pflüger's Arch.*, **50**, 1891 (98); **77**, 1899 (425).

[8] A preliminary account of this is given by Osborne, T. B., and Mendel, L. B., *Proc. Soc. Exper. Biol. Med.*, **18**, 1921 (167).

VARIATION AND INHERITANCE IN SIZE IN TRYPANOSOMA LEWISI

II. The Effects of Growing "Pure Lines"[1] in Different Vertebrate and Invertebrate Hosts and a Study of Size and Variation in Infections Occurring in Nature

By W. H. Taliaferro

Department of Medical Zoölogy of the School of Hygiene and Public Health, Johns Hopkins University

Communicated by R. Pearl, May 2, 1921

In our first report[2] we demonstrated that Trypanosoma lewisi reaches an adult stage in the course of the infection in the rat in about 25 days, and once this stage is reached there is practically no division or growth. In consequence of the elimination of growth factors from the computations it was found that the coefficient of variation in the "pure line" was very low if it was computed during this adult period. The questions which now arise are: (1) Does growing the same "pure line" in different vertebrate hosts cause significant differences in the mean or the coefficient of variation? (2) Does passage of the "pure line" through the invertebrate host cause any significant differences in these constants? Our attention will be directed especially toward the coefficient of variation as this will give us an opportunity to ascertain whether or not there is a splitting up of the "pure line" into heritably diverse lines following such passage through the invertebrate host. With this background our final question is: (3) Does an infection occurring in nature consist of a large number of "pure lines" such as have been described in free-living protozoa, which differ among themselves but are *per se* constant in size?

It is to be emphasized at this point that in all of the following work the measurements and computations were made on or after the 30th day of the blood infection. As the adult stage is always reached by the 25th day this is well into the adult period and effectually eliminates growth factors. The six measurements of size used in this work have already been given in our first report (see figure 1). They consist of the following distances: (1) Posterior end *to* parabasal body, (2) parabasal body *to* nucleus, (3) nucleus *to* anterior end, (4) anterior end *to* end of flagellum, (5) total length which includes measurements 1–4, and (6) width.

Effect of Growing the Same "pure line" in Different Vertebrate Hosts.— The twoconstants in which we are interested in this connection are the size as indicated by the mean and the variability as indicated by the coefficient of variation. In all of the experiments the constants were worked out for all six measurements but in this brief report we will only deal with those for total length. Let us first consider the mean. The questions are: Does growing the same pure line in different *individuals* of the same species of rat produce significant differences in size? Does growing the same "pure line" in different *species* of rats produce any greater differences in size than in the first case of individuals belonging to the same species? It would have been interesting to grow the "pure line" in other vertebrates as well as in rats but T. lewisi is not normally infective to any other vertebrate. To test the first of these questions we used individuals of the albino rat and for the second question we used individuals of the black and the Norway rat. At the outset it may be definitely said that the differences which were obtained on growing the "pure line" in different species of rats were in no way more significant than the differences obtained by growing the "pure line" in different individuals of the same species. The greatest difference in mean size[3] was obtained when the same "pure line" was grown in the albino rats 46 and 90. In the former the mean length was 32.080 ± .052 and in the latter 30.769 ± .057, which gives us a difference of 1.311 ± .077. The mean lengths obtained in twelve other experiments lie between these limits. While a difference of this character, which is 16.8 times its probable error, is undoubtedly significant the question immediately arises as to whether such a difference is due to differences in the blood of different rats or to a personal variation in making the measurements. A definite answer cannot be given to this question at the present time although experiments are being carried out to ascertain, if possible, which is the determining factor. As, however, this is the greatest difference obtained in twelve experiments, we may say that growing the same "pure line" in different rats probably causes significant differences in mean size, but in any case these differences are small. Even if these differences are significant there is no evidence, at present, that they are due to inherited diversities, but are simply due to differences in environment.

In regard to variability, the chief thing of interest is the fact that passage of the "pure line" from rat to rat by blood inoculation has no effect on the coefficient of variation. Take, for example, the following experiment. The coefficient of variation for total length in the "pure line" as grown in rat 116 was 2.77 ± .13%. After four passages through white rats this line was grown in rat 204 where the coefficient of variation was found to be 2.60 ± .12%. The difference between these two coefficients is .17 ± .18% which certainly is not significant. Several other experiments gave similar results and an experiment is being carried out now in which the "pure line" is undergoing numerous transfers from one rat to another. The fact that passage of the pure line through rats has no effect on the coefficient of variation is in marked contrast to conditions found after passage through the invertebrate host.

Effect of Passing the "pure line" through the Invertebrate Host.—While T. lewisi can be experimentally transmitted by several species of ectoparasites the natural insect vectors are the rat fleas, Xenopsylla cheopis and Ceratophyllus fasciatus. In working out the life history of certain species of trypanosomes some of the earlier investigators described what they considered to be sexual phases in the life history of the parasite. Practically all of these observations are open to entirely different interpretations, so that at the present time, although a number of protozoölogists feel that there is a sexual phase in the life cycle, no such phase has been demonstrated. This is notably true in the classic researches of Minchin and Thomson[4] on the life history of T. lewisi in the rat flea. Although they looked carefully for conjugation they found no evidence of it.

While there is no evidence of sexual phenomena in the trypanosomes during passage through the invertebrate host, there is evidence that such passage exerts a profound effect on such things as acquired physiological characteristics. Gonder[5] for example found that arsenic fastness was transmitted from rat to rat but was lost by passage through the louse. Miss Robertson[6] found that strains of T. gambiense showed marked changes in their characteristics after passage through the tsetse-fly. This led this author to say, "It seems clear that the cycle in the fly as a whole, whether conjugation occurs or not, has much of the biological significance of the process."

As we have seen passage of the same "pure line" through a number of rats does not increase the variability of the "pure line." Passage through the flea, on the other hand, *invariably* increases the variability. These experiments were carried out in the following manner. Fleas which were carefully raised in the laboratory and which were known to be free from infection were allowed to bite animals infected with a given "pure line." After about a week (during which time the trypanosomes were undergoing their development in the flea) other rats were infected either by teasing up a single flea and injecting it intraperitoneally or by allowing the rat to lick up the moist feces, which is the natural mode of infection. The trypanosomes in the rat which was infected from the flea were measured on the 30th day of the blood infection. Six such experiments have been successfully carried out. The following is a fair example of all of them. The pure line in rat 105 showed a coefficient of variation of $2.80 \pm .13\%$ for total length. After passage through a single specimen of X. cheopis it was measured in rat 163. Here the coefficient of variation increased to $5.24 \pm .25\%$, a difference of $2.44 \pm .28\%$. As we have seen above, no such increase in variability has been observed during passage of the "pure line" from rat to rat. From these experiments we must conclude that the "pure line" breaks up during passage through the invertebrate host. This is analogous to the results of Jennings[7] in Paramecium after conjugation and in a recent publication Miss

Erdmann[8] believes that she has demonstrated the same type of phenomena in Paramecium after endomixis. While the increase in variability is probably due to some nuclear phenomena during the life cycle of the trypanosome in the flea, we have no method of judging, from the present data on the subject, whether it is a sexual or a reorganization process. Minchin and Thomson[4] have suggested that the peculiar effects produced by passage through the invertebrate host are connected with the transformation of the trypanosome into a crithidial stage and the reverse process, both of which always take place in the life cycle in the invertebrate host.

It is conceivable that passage of a "pure line" through the invertebrate host might cause significant changes in the mean size as well as in the variability. All of our experiments have shown this not to be true. It is true that after the "pure line" is passed through a flea there is often a difference in the mean which is significant from a statistical standpoint, but as these differences are never greater than the maximum difference given above for growing the "pure line" in different rats, we cannot ascribe the difference to any peculiar effect of the flea. It is probably due simply to the fact that the trypanosomes are measured in different rats.

Size and Variability in "wild"[1] *Infections Occurring in Nature.*—A number of naturally infected rats were collected from around Baltimore and Washington. Up to the present time measurements have been made of specimens from ten of these infections. While these "wild" infections do show differences in their means and a higher coefficient of variation on the average than the pure lines, one is struck with the constancy of the means and the low variability. The longest mean length (rat 221) obtained is 32.503 ± .060 and the shortest (rat 411) 29.093 ± .062 with a difference of 3.410 ± .087. It can be seen that the difference between the longest "wild" infection and the shortest is comparatively great and that the difference is much greater than is obtained by growing a "pure line" in different rats. The least variable "wild" infection (rat 413) showed a coefficient of variation of only 2.125 ± .101. This is slightly lower than any of the coefficients of the "pure lines" in the laboratory. The most variable "wild" infection (rat 65) showed a coefficient variation of 4.583 ± .223 The difference between the least and the most variable "wild" infection is 2.45 ± 24. The remainder of the "wild" infection measure fall between the figures given here, both as regards their means and coefficients of variation.

From these results we must conclude that while the "wild" infections occurring in nature show comparatively small differences in size, these differences are greater than can be explained by the fact that the infections are occurring in different rats. We must also conclude that most "wild" infections consist either of very few "pure lines" or a larger number which differ *inter se* by very minute differences. In some cases, as in that of rat 413, we are almost forced to believe that a given "wild" infection is actually a "pure line." A possible explanation of these facts suggests itself when we consider the mechanism of infection with T. lewisi. As stated above the rat receives the infection from the flea. As the infected flea is feeding it deposits its feces, which contain the infective forms of the trypanosomes, on the fur of the rat. The rat then becomes infected by licking up the *moist* feces containing the infective forms. In nature it is probable that a given rat receives its infection from a single flea. This flea deposits a number of infective forms. The chances are that most of these will be killed by drying and that only a few will actually start the infection in the rat. This being so, the variability of the infections simply reflect the number of infective forms of the trypanosomes that actually start growth in the rat. In fact it is conceivable that a number of the "wild" infections are actually "pure lines."

At the present time the author is of the opinion that if we consider all the infections in nature collectively we can say that there are a large number of "pure lines" which differ among themselves but which (as is shown by work in the laboratory) are *per se* constant in size. Due to the peculiar way in which rats receive their infections, however, we find that any given infection consists of only a very few of these "pure lines."

Conclusions.—At the present phase of the work the following conclusions may be drawn: While growing the same "pure line" in different rats may cause significant differences in the mean size, these differences are small. The differences in the mean are never greater when the "pure line" is grown in different species of rats than when it is grown in different individuals of the same species. Passage of the "pure line" from rat to rat is not followed by any significant changes in the coefficient of variation. In marked contrast to this it is found that passage of the "pure line" through the flea is invariably followed by a significant increase in the coefficient of variation which is interpreted as showing that the "pure line" breaks up into heritably diverse lines following such passage. Although passage through the flea has such a marked effect on the variability it has no significant effect on the mean size. "Wild" infections occurring in nature exhibit differences in their mean sizes which are greater than can be explained by the fact that they occur in different rats. The coefficients of variation of "wild" infections are, on the average, higher than for the "pure line." The lowest of these coefficients, however, is no greater than the coefficient of variation for the "pure line" and even the highest can be explained on the assumption that the "wild" infection consists of only a very few "pure lines." It seems probable that there occur, in nature as a whole, a large number of "pure lines" which differ among themselves but are *per se* constant in size, but, due to the peculiar way in which rats receive their infections, a given infection consists of only a very few of these lines. In fact, a large number of "wild" infections are probably actual "pure lines."

---

[1] Throughout this work the term "pure line infection" is used to designate an infection, the trypanosomes of which have all arisen from a single organism. A given "pure line" may either have been started from a single trypanosome, or it may have been subinoculated from such an infection. The term "wild infection" designates an infection as found in nature.

[2] These PROCEEDINGS, **7,** 1921 (138–143).

[3] The mean sizes in this report are all in microns.

[4] Minchin, E. A., and Thomson, J. D., *Quart. J. Microsc. Sci.*, **60,** *N. S.*, 1915 (463–692).

[5] Gondor, R., *Centralbl. Bakt., etc.,* I abt., Originale, **612,** 1911 (102–113).

[6] Robertson, M., *Proc. Roy. Soc.*, (B) **85,** 1912 (241–248).

[7] Jennings, H. S., *J. Exper. Zool.*, **11,** 1911 (1–134); *Ibid.*, **14,** 1913 (270–391).

[8] Erdmann, R., *Archiv. Entwicklungsmech. Organ.*, **46,** 1920 (85–148).

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*NOTE ON MOVING EQUILIBRA**

BY ALFRED J. LOTKA
BROOKLYN, N. Y.

Communicated by R. Pearl, March 12, 1921

A number of previous publications[1] have been devoted to the study, from various angles, of a material system evolving in accordance with a system of differential equations

$$dX_i\,/\,dt = F_i(X_1, X_2,\dots; A_1, A_2,\dots; P; Q) \qquad (1)$$

where the X's denote the masses of the several components of which the system is built up; the A's are parameters introduced by any equations of constraint to which the X's may be subject;[2] the parameters P include geometrical constraints (volume, area, topography) and also other quantities serving to define the state of the system (temperature, etc.). The Q's define the character of the several components or species.

The discussion has hitherto been restricted to the case that the parameters A, P, Q remained constant during the transformations taken in view. A complete discussion of the evolution of systems of the kind referred to must include also the consideration of changes in these parameters.

Such changes may be grouped under three heads:

1. Changes of a perfectly general character. A study of these would resolve itself into a discussion, on a general basis, of a system of differential equations of the form

$$dX_i\,/\,dt = F_i(X_1, X_2,\dots,t) \qquad (2)$$

It is not intended to enter here into this perfectly general case. It will suffice to point to the purely mathematical literature on the subject.[3]

2. Changes in the parameters A, P, Q proceeding very slowly as compared with the speed of readjustment of the X's. In this case the system, after passing through a "transient" state, ultimately settles down to a "moving equilibrium." The study of this case therefore comprises two phases, which may with advantage be taken up separately. The phase of the moving equilibrium is of particular interest, since such moving equilibria play an important rôle in the evolution of physical systems, as pointed out years ago by Herbert Spencer.[4]

3. We may study the effect of a change in a parameter A, P or Q upon the equilibrium of the system alone, irrespective of the process by which that equilibrium is reached. It is in this case immaterial whether the change is slow or rapid. Into this division of the subject fall such relation as the principle of Le Chatelier, the thermodynamic laws of equilibrium and the "reciprocal relations" of statistical mechanics.

The application of some of these principles to biological and social systems has been essayed, but it cannot be said that the rigor of the attempts thus made is satisfactory. It would therefore be desirable to go over the ground and consolidate it. An effort in this direction is taken in view as part of the plan into which the present contribution is fitted.[5]

Of the general field outlined above, the portion to which we shall now give our attention is that of moving equilibria.

Our fundamental system of equations we shall, for our present purpose, write in the form (2). Furthermore, merely in order to simplify our notation, we will restrict the number of dependent variables to two, which, to avoid subscripts, we will denote by X, Y. We have, then

$$dX/dt = F_1(X, Y, t), \qquad dY/dt = F_2(X, Y, t). \tag{3}$$

We adopt a method of successive approximations. Since the system is near equilibrium, we write for our first approximation

$$0 = F_1(X, Y, t), \qquad 0 = F_2(X, Y, t). \tag{4}$$

Solving for X and Y we then have

$$X_1 = \varphi_1(t), \qquad X_1 = \psi_1(t). \tag{5}$$

whence by differentiation

$$dX_1/dt = X_1' = \varphi'_1(t), \qquad dY_1/dt = Y'_1 = \psi'_1(t). \tag{6}$$

Proceeding to a second approximation, we substitute (6) in (3).

$$\varphi'_1(t) = F_1, \qquad \psi'_1(t) = F_2. \tag{7}$$

Solving again for X, Y,

$$X_2 = \varphi_2(t), \qquad Y_2 = \psi_2(t.) \tag{8}$$

We may again differentiate,

$$X'_2 = \varphi'_2(t), \qquad Y_2 = \psi'_2(t). \tag{9}$$

and, substituting as before, we obtain a third approximation. We continue this process as far as may be desired, and finally obtain for the *n th* approximation

$$X_n = \varphi_n(t), \qquad Y_n = \psi_n(t). \tag{10}$$

If the functions φ_n, ψ_n, and φ'_n, ψ'_n, obtained by these successive approximations tend toward a limit as n is increased indefinitely, then it can easily be shown that (10) is a solution of (3).

It seems to be somewhat difficult to give in general terms the conditions for this convergence of successive approximations toward a limit. However, the following example will show how in individual cases the question regarding this convergence can sometimes be readily answered.

For our example we take the case of radioactive equilibrium. We have a chain of transformations

$$A \longrightarrow B \longrightarrow C \longrightarrow D \tag{11}$$

We will denote the masses of A, B, C, D at time t, respectively, by U, X, Y, Z, and their values at time $t=0$ by the same letters with the subscript zero.

We may if we choose (this is a purely arbitrary matter) pick out the substances B and C for our evolving system, and look upon A and D as external factors influencing the system. The system composed of B and C is then subject to an equation of constraint

$$X + Y = X_0 + Y_0 + U_0 - U + Z_0 - Z \tag{12}$$

in which the quantities appearing in the right hand member are parameters of the class denoted above in equation (1) by A, that is to say, parameters introduced by the equation of constraint. Four of them are constants, the other two are functions of t. Of these last two one, namely U, will appear in the equations representing the course of evolution of the system composed of B and C. These equations, according to the well-known laws of radioactive transformations are

$$dX/dt = aU - bX, \qquad dY/dt = bX - cY. \tag{13}$$

where a, b, c are constants. U, on the other hand, is a function of the time, namely

$$U = U_0 e^{-at} \tag{14}$$

For our first approximation we put, then,

$$dX/dt = 0 = aU_0 e^{-at} - bX_1 \qquad dY/dt = 0 = bX_1 - cY_1 \tag{15}$$

whence

$$X_1 = a/b.\ U_0 e^{-at}, \qquad Y_1 = a/c.\ U_0 e^{-at}, \tag{16}$$

and therefore

$$X'_1 = -a^2/b.\ U_0 e^{-at}, \qquad Y'_1 = -a^2/c.\ U_0 e^{-at}. \tag{17}$$

The second approximation now gives

$$dX/dt = X'_1 = -a^2/b.\ U_0 e^{-at} = aU_0 e^{-at} - bX_2 \tag{18}$$

$$dY/dt - Y'_1 = -a^2/c.\ U_0 e^{-at} = bX_2 - cY_2 \tag{19}$$

Whence

$$X_2 = a/b.\ U_0 e^{-at}\ (1 + a/b) \tag{20}$$

$$Y_2 = a/c.\ U_0 e^{-at}\ (1 + a/b + a/c) \tag{21}$$

Continuing this process we find, ultimately,

$$X_n = a/b\ U_0 e^{-at}\ (1 + a/b + a^2/b^2 + \dots) \qquad (22)$$

$$Y_n = a/c\ U_0 e^{-at}\ (1 + a/b + a/c + a^2/b^2 + a^2/bc + a^2/c^2 + \dots) \qquad (23)$$

In this example the condition for the convergence of the successive approximations is immediately apparent. We must have

$$a/b < 1, \qquad a/c < 1; \qquad (24)$$

that is to say, the parent substance A must have a smaller decay constant than any of the succeeding members of the series. For obvious reasons this condition is always satisfied in natural radioactive mixtures.[6]

The series (22), (23) bring out the relation between the uncorrected equilibrium, as commonly computed on the assumption of constancy of mass of the parent substance, and the true equilibrium. The first-mentioned (for which Rutherford has suggested the term "secular equilibrium") is represented by the first term of the series. As Rutherford points out, the error of the first approximation, i.e. the difference between the secular and the true equilibrium, amounts, in some cases, to nearly 1% though in others the error is quite negligible.

The series are easily summed, and then lead to the well-known expressions obtained by other methods (for the equations of radioactive change are readily integrable in finite terms, while the method here developed is applicable also in more refractory cases).

The case of radioactive equilibrium was here selected as an illustration, primarily because the functions involved are known and of simple form. But the same example will serve very aptly to illustrate also some other points.

In the first place we observe that moving equilibria might be divided into three classes, according as their progress is determined by a change in the P's, the Q's or the A's. As has been shown, the radioactive equilibrium is of the type in which the pace is set by a parameter of the class A, namely *the mass of one of the links in the chain*, which thus acts as a brake, or a *limiting factor* checking the series of transformations. Such limiting factors play an important rôle also in the highly complex network of interlocking cycles upon which the continuance of abundant life upon the earth depends. For life processes are energy transformation processes carried out by the agency of material energy transformers. Such transformers, if they are to work continuously and indefinitely must perforce work in closed transformation chains or cycles (such as the cycle $CO_2 \rightarrow$ Plant $\rightarrow$ Animal $\rightarrow CO_2$). The moving equilibria engendered in such systems of cycles by a slow change in a limiting factor, in a parameter of class A, invite further study. The influence of man upon the world's events seems to have been largely to accelerate the circulation of matter and energy through such cycles, either by "enlarging the wheel", i.e., increasing the mass taking part in certain cycles, or else by causing it to "spin faster," i.e., increasing the velocity of the circulation, decreasing the time required for a given mass to complete the cycle. In either case he has increased the energy turn-over per unit of time. Whether, in this he has been unconsciously fulfilling one of those laws of nature according to which certain quantities tend toward a maximum, is a question well deserving of our attention.

* Papers from the Department of Biometry and Vital Statistics, School of Hygiene and Public Health, Johns Hopkins University, No. 44.

[1] Lotka, A. J., *Physic. Rev.*, **24**, 1912 (235–238); *J. Washington Acad. Sci.*, **2**, 1912 (2, 49, 66); *Science Progress*, **55**, 1920 (406–417); *Proc. Am. Acad. Arts Sci.*, **55**, 1920 (237–153); these PROCEEDINGS, *Sci.* **6**, 1910 (410–415).

[2] *Proc. Amer. Acad.*, loc cit., p. 142.

[3] See for example Picard, *Traité d'Analyse;* H. Bateman, *Differential Equations*, 1918, p. 245.

[4] Spencer, *First Principles*, Chapter XXII; Winiarskie, "Essai sur la Mécanique Sociale," *Revue Philosophique*, **44**, 1900 (113).

[5] At the time of reading proof this project is partially realized. A discussion of the applicability of the Le Chatelier principle to systems of the general character here considered will appear in a forthcoming issue of the *Proceedings of the American Academy of Arts and Sciences*

[6] Lotka, A. J., *London, Phil. Mag.*, Aug., 1911, p. 353.

A FORMULA FOR THE VISCOSITY OF LIQUIDS[1]

By H. B. PHILLIPS

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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1. In this paper I obtain for the viscosity of a liquid the formula

$$\eta = \frac{n\dot{N}h}{2M(v-\delta)}, \qquad (1)$$

where N is the number of molecules in a mol, h is Planck's constant, M is the molecular weight of the liquid in the gas phase, v its volume per gram, and n an integer. The quantity δ is the co-volume as used in the equation of state of Keyes[1]

$$p = \frac{RT}{v-\delta} - \frac{A}{(v-l)^2}, \qquad (2)$$

In all the cases to which I have applied the formula, $n = 6$ and so (1) takes the form

$$\eta(v-\delta) = 3Nh/M \qquad (3)$$

It is to be noted that $3N/M$ is the number of translational degrees of freedom of the molecules in the volume v of the liquid.

2. To prove equation (1), let x, y, z be rectangular coördinates and suppose the liquid to flow parallel to the x-axis in such a way that, u_0 being

the velocity at the point (x, y, z),

$$\partial u_0/\partial z = 1. \tag{4}$$

Let the components of velocity of a molecule parallel to the x and z axes be u, w respectively. When a molecule of mass m moves in the positive direction of the z-axis across the xy-plane, the x-component of momentum transferred across that plane is mu. When it crosses in the negative direction the transfer is $-mu$. Under these conditions the viscosity η of the liquid is defined as the total x-component of momentum transferred across unit area of the xy-plane in unit time.[2] To find the viscosity we therefore find the number of molecules of each velocity crossing unit area and multiply by the average momentum transferred by each.

Let $f(w)\,dw$ be the number of molecules per cubic centimeter with components of velocity parallel to the z-axis between w and $w + dw$. If the molecules did not influence each other, the number of these that would cross unit area of the xy-plane in unit time would be

$$w\,f(w)\,dw. \tag{5}$$

Because of the interference of molecules with each other, this must be replaced by

$$\frac{v}{v-\delta}\,w\,f(w)\,dw. \tag{6}$$

To see this we note that if the molecules moved independently, the z-component of momentum transferred across unit area of the xy-plane in both directions in unit time would be the pressure

$$RT/v. \tag{7}$$

From equation (2), in the real liquid this is replaced by

$$RT/(v-\delta) \tag{8}$$

Now the mass of the molecule and the temperature (average kinetic energy) are the same in both cases. Hence the increased momentum (8) compared with (7), can only be due to an increase in the number of molecules crossing in unit time. Since the same law of distribution of velocities (Maxwell's Law) is assumed in both cases, the ratio in which the numbers are increased must be the same $v/(v-\delta)$ for all velocities. Thus the number of molecules with velocity components between w and $w + dw$ crossing unit area of the xy-plane in unit time is given by equation (6).

The velocity u of a given molecule parallel to the x-axis will not always be equal to the average velocity u_0 of the liquid at that point. There will, however, be a series of instants (which we may call collisions) when u will be equal to u_0. I assume that the interval between two such instants will be a half period (interval between libration limits)[3] in the sense of the quantum theory, and that the molecules can be treated as moving with constant velocity between consecutive collisions. Suppose then a molecule reaching the xy-plane has been moving a time, t, since its last collision.

From (4) its momentum parallel to the x-axis due to the motion of the liquid at that point will be $m\ w\ t$.

From (6) the total x-component of momentum transferred in the positive direction across unit area of the xy-plane in unit time will then be

$$\frac{v}{v-\delta}\int_0^\infty m\ w^2 t f(w)\, dw.$$

There will be an equal transfer due to molecules moving in the negative direction. Hence

$$\eta = \frac{v}{v-\delta}\int_0^\infty 2m\, w^2\, t f(w) dw. \tag{9}$$

The time t can be considered as the interval between collisions at times[4] t_0, t_1. Since the molecule is assumed to move with constant velocity between collisions,

$$2m\, w^2\, t = 2\int_{t_0}^{t_1} m\, w^2\, dt = nh, \tag{10}$$

where, according to Sommerfeld's theory, n is an integer. Also

$$v\int_0^\infty f(w)\, dw = N/2M, \tag{11}$$

since it is the number of molecules per gram for which w is positive. Combining (9), (10), and (11), we get (1) which was to be proved.

3. Owing to lack of data on the equation of state, the only substances on which the formula can at present be tested are carbon dioxide, ether, and mercury. Using the values of N and h given by Birge,[5]

$$3Nh = 3(6.0594)(6.5543)10^{-4} = .011914,$$

and so equation (3) takes the form

$$\eta(v-\delta) = .011914/M. \tag{12}$$

Values of δ for carbon dioxide and ether,[6] tables I and II, were supplied by Professor Keyes. At low temperatures the measured viscosities and those calculated by equation (12) do not differ by more than the experimental error. In case of carbon dioxide the temperature 30° is too near the critical point ($t=31°$) for satisfactory use of the equation of state. Above 10° the calculated viscosity of ether is too large, the difference increasing with the temperature. This may be due to the fact that ether is a complex of more than one type of molecule. The equation of state was determined on the assumption that each liquid molecule is formed by the combination of two gas molecules. This may be substantially true at 10° and not at 100°.

To obtain values of $v-\delta$ for mercury, I make use of the fact that monatomic substances seem to have constant co-volumes[7] δ. Since mercury is monatomic in the gas phase, I assume that δ is constant or nearly constant in the liquid phase. Also the term

$$RT/(v-\delta)$$

in case of mercury is very large (more than 15000 atmospheres). Hence at atmospheric pressures we may neglect p and so write (2) in the form

$$\sqrt{\frac{V-\delta}{T}}=c\ (v-l)$$

where c is constant. Differentiating this equation with respect to v and then with respect to T, assuming δ constant, we get

$$(v-\delta)^2\left(3\frac{dv}{dt}+2T\frac{d^2v}{dt^2}\right)-2T(v-\delta)\left(\frac{dv}{dt}\right)^2-T^2\left(\frac{dv}{dt}\right)^3=0,$$

whence

$$V-\delta=T\frac{dv}{dt}\left[\frac{\frac{dv}{dt}+\sqrt{4\left(\frac{dv}{dt}\right)^2+2T\frac{d^2v}{dt^2}}}{3\frac{dv}{dt}+2T\frac{d^2v}{dt^2}}\right]. \qquad (13)$$

According to Callendar and Moss[8]

$$\frac{V}{V_0}=1+1805553\left(\frac{t}{100}\right)10^{-8}+12444\left(\frac{t}{100}\right)^2 10^{-8}+2539\left(\frac{t}{100}\right)^3 10^{-8} \qquad (14)$$

is the ratio of the volume of mercury at $t°$ to its volume at $0°$. The values of $v-\delta$ in table III are obtained by calculating the derivatives of v from (14) and substituting in (13). Below 100° the difference of the calculated viscosities and those measured is not greater than the difference between the values obtained by different observers. At higher temperatures the calculated viscosities are consistently smaller than those measured. The observations, however, differ greatly, owing probably to oxidation of the mercury in contact with the air. The result may be to increase the viscosity by almost any amount. This work should be repeated with mercury of the greatest purity in a vacuum or in contact with some inactive gas.

TABLE I

CARBON DIOXIDE

$$\log\ \delta=.09780-\frac{0.1978}{v},\quad \eta(v-\delta)=.0002708$$

TEMP.	PRESSURE	v	$v-\delta$	$\eta \times 10^6$ CAL.	$\eta \times 10^6$ OBS.	OBSERVER
5	Sat.	1.113	.278	974	925	Warburg and Babo[9]
10	"	1.168	.317	854	852	
15	"	1.228	.360	751	784	
20	"	1.306	.419	646	712	
20	59 at.	1.291	.408	664	697	Phillips[10]
20	72	1.224	.358	756	771	
20	83	1.189	.332	816	823	
30	72	1.566	.627	432	458	
30	90	1.339	.445	608	643	
30	104	1.274	.395	686	733	

TABLE II

ETHER

$$\log\ \delta = .48959 - \frac{.50981}{v},\quad \eta(v-\delta) = .0001608$$

TEMP.	v	$v-\delta$	$\eta \times 10^6$ CAL.	$\eta \times 10^6$ OBS.	OBSERVER
0°	1.3583	.0573	2806	2860	Thorpe and Roger[11]
10	1.3797	.0612	2626	2585	
20	1.4016	.0663	2425	2345	
30	1.4247	.0703	2287	2120	
30	1.4247	.0703	2287	2134	Pound[12]
30	1.4247	.0703	2287	2150	Heydweiller[13]
40	1.4506	.0761	2113	1970	
50	1.4784	.0828	1942	1810	
60	1.5019	.0888	1811	1666	
70	1.5309	.0961	1673	1528	
80	1.5620	.1058	1519	1400	
90	1.6000	.1177	1366	1284	
100	1.6380	.1302	1235	1177	

TABLE III

MERCURY

$$\eta(v-\delta) = .00005935$$

TEMP.	$v-\delta$	η-CAL.	η-OBS.	$\eta(v-\delta)$	OBSERVER
−18.1	.0033282	.01784	.01836	.00006111	Koch[14]
0	.0035606	.01667	.01661	.00005914	Plüss[15]
0	.0035606	.01667	.01688	.00006010	Koch
10	.0036885	.01609	.01577	.00005817	Umani
10.1	.0036897	.01609	.01620	.00005975	Koch
14.95	.0037538	.01583	.01571	.00005898	Benard[16]
15.1	.0037557	.01581	.01575	.00005914	"
15.7	.0037635	.01578	.01570	.00005909	"
15.9	.0037661	.01577	.01583	.00005961	"
15.95	.0037668	.01577	.01581	.00005954	"
16.1	.0037687	.01576	.01563	.00005889	"
16.2	.0037700	.01575	.01563	.00005892	"
16.5	.0037717	.01574	.01572	.00005929	"
18.76	.0038006	.01562	.01558	.00005921	Plüss
20	.0038165	.01555	.01547	.00005904	"
20	.0038165	.01555	.01579	.00006026	Schweidler
34.05	.0039967	.01485	.01476	.00005899	Plüss
40	.0040732	.01457	.01483	.00006041	Schweidler
62.9	.0043682	.01359	.01360	.00005941	Plüss
79.4	.0045817	.01295	.01299	.00005952	"
97.95	.0048225	.01230	.01263	.00006091	"
99	.0048362	.01227	.01227	.00005934	Koch
124	.0051589	.01151	.01171	.00006041	"
137.5	.0053304	.01114	.01167	.00006221	Plüss
154	.0055456	.01070	.01092	.00006056	Koch
			av.	.00005968	

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[1] F. G. Keyes, "A New Equation of Continuity," *Proc. Nat. Acad. Sci.*, **3,** 1917 (323–330).

[2] Jeans, *Dynamical Theory of Gases,* p. 247.

[3] Adams, "The Quantum Theory," *Bull. Nat. Res. Council,* **1,** No. 5, Oct., 1920.

[4] Jeans, *loc. cit.,* p. 246.

[5] Ithaca, *Physic. Rev.,* **14,** p. 368.

[6] Wm. A. Felsing (Thesis), Technology Press, June, 1918.

[7] Keyes, *loc. cit.*

[8] *Proc. R. Soc. London,* **84A,** p. 595.

[9] Leipzig, *Ann. Physik,* **17A,** 1882 (418).

[10] *Proc. R. Soc. London,* **87A,** pp. 556–57.

[11] *Phil. Trans. R. Soc.,* **185A,** p. 572.

[12] *J. Chem. Soc. London,* **99,** p. 708.

[13] *Ann. Physik,* **59,** p. 200.

[14] *Landolt Bornstein* (1912).

[15] *Z. Anorg. Chem.,* **93,** p. 18.

[16] Brillouin, *Leçons sur la Viscosité,* pp. 152–159.

THE ANGULAR DIAMETER OF ALPHA BOOTIS BY THE INTERFEROMETER

BY F. G. PEASE

MOUNT WILSON OBSERVATORY, CARNEGIE INSTITUTION OF WASHINGTON

Communicated by G. E. Hale, May 2, 1921

Since the measurement of the diameter of Betelgeuse[1] in December, 1920, observations for the determination of stellar diameters have been continued with the 20-foot interferometer attached to the upper end of the 100-inch Hooker reflector.

Several definite results have been obtained, and others, less definite because of poor observing conditions, have been partially checked by comparison with stars whose diameters are known to be too small for measurement with this instrument. Let the visibility of the zero fringes for any particular separation of the mirrors be reduced by seeing which is poor as compared with that on a fine night; it is then hopeless to make final measures. If, however, the interferometer is turned to a neighboring check star and the fringes are seen, and then turned to the star under consideration and they do not appear, the observer is justified in saying that there is a definite decrease in visibility.

The data thus far obtained are as follows: For α Tauri (Aldeberan) fringes of gradually decreasing visibility have been observed at 13, 14.5 and 19 feet. Further observations are necessary between the last two positions to determine whether the fringes vanish between these points or whether a longer beam will be necessary to obtain a measure of the diameter. If the fringes vanish around 16–18 feet, as observations under poor conditions lead one to suspect they will, then the fringes observed at 19 feet must lie beyond the point of disappearance, on the ascending branch of the visibility curve where it rises toward the second maximum.

For β Geminorum (Pollux) fringes have been observed at separations of 14.5, 16, and 19 feet, the visibility being definitely reduced for these distances of the mirrors, although at 19 feet it is still considerable. A longer beam will therefore be required for the measurement of the star's diameter.

For α Cetione observation at 13 feet in variable seeing showed a reduction in visibility of the fringes when compared with that of the check star.

For α Bootis (Arcturus) observations in February and March, with poor seeing, indicated a decided reduction in the visibility of the fringes with increasing separation of the mirrors. On April 15, a series of measures was made under conditions which were excellent, as indicated by the strong contrast of the zero fringes and by diffraction rings surrounding the central disk of the star images.

At 16 feet separation of the mirrors the visibility of the fringes was estimated at 20 to 25 per cent of the zero fringes; at 17.5 feet, 15–20 per cent, certainly less conspicuous than at 16 feet. At 19.0 feet the estimate was 5 per cent, the contrast being very flat, but the fringes were picked up with certainty as the path difference was varied with the optical wedge. At 18.25 feet, the result was 15 per cent, certainly greater than at 19.0 feet; and finally at 19.5 feet no fringes whatever could be observed from the interferometer mirrors. There was no bright star in the immediate neighborhood that could be used as a check, but the instrument was known to be in adjustment for the final measure, from the certainty with which the previous settings had been made.

The separation of the mirrors for disappearance of the fringes may therefore be placed at 19.5 ± 0.5 feet, although observations with a longer beam with which the second maximum could be observed, might possibly increase this value a trifle. By disregarding any possible darkening at the limb, the result will serve to indicate the maximum diameter of the star. The spectrum of α Bootis is Ko; its approximate effective wave-length may therefore be taken at 5.6×10^{-5} cm., about halfway between that of the sun (Go) and that assumed for α Orionis (Ma).

With this value of λ and a distance of 19.5 feet (594 cm.), the angular diameter of α Bootis is 0.0237. It is interesting to note the close agreement of this figure with the estimated values of Eddington (0.020) and Russell (0.019), and with Hertzsprung's value (0.026), calculated on the basis of the measured diameter of α Orionis (0.045).

Results for the parallax of α Bootis are as follows: Yale, 0.074; Flint, 0.095; Yerkes, 0. 100; Adams, spectroscopic, 0.158; the weighted mean is 0.116. The distance of Arcturus is therefore of the order of 1.65×10^{14} miles (2.65×10^{14} km.), and its linear diameter 19×10^{6} miles (30.6×10^{6} km.), or about 22 times the diameter of the sun.

All settings of the mirrors have thus far been made by hand, and considerable time is consumed between measures. Two screws are now being mounted on the beam, driven by a single motor, which for any separation will keep the outer mirrors equidistant from the fixed inner mirrors, and thus greatly facilitate the operation of the interferometer.

[1] These PROCEEDINGS, **7,** 1921 (143–146).

CHANGES OBSERVED IN THE CRAB NEBULA IN TAURUS

BY JOHN C. DUNCAN

MOUNT WILSON OBSERVATORY, CARNEGIE INSTITUTION OF WASHINGTON

Communicated by George E. Hale. Read before the Academy, April 26, 1921

Changes in the structure of the Crab nebula (N. G. C. 1952, M. 1) have recently been detected by Lampland[1] from a comparison of seventeen photographs made with the 40-inch Lowell reflector during a period of eight years. An excellent negative of this nebula had been made by Ritchey with the 60-inch Mount Wilson reflector, 1909, October 13, using a Seed 27 plate and giving an exposure of three hours with the aperture stopped down to 54 inches. For comparison with this plate, the writer made a negative with the same instrument, 1921, April 7, using full aperture and a Seed 30 plate and giving an exposure of 1 hour 20 minutes. Though the exposure of this plate was necessarily shorter than that of Ritchey's because of the proximity of the object to the sun, the larger aperture and the faster plate made the photograph comparable in density with the earlier one.

The two plates were compared in the stereocomparator, and it was seen at once that changes in the relative luminosity of different parts of the nebula had occurred, particularly in the bright region north-west of the center; and also that certain filaments and condensations had apparently moved, the motion in general being away from the center, though not always exactly along a radius.

Measures in two coördinates were made with the micrometer of the stereocomparator upon twelve nebulous condensations and thirteen comparison stars. The nebulous points chosen are sufficiently symmetrical to permit measures of about equal accuracy in the two directions. They were chosen in the outer, faint parts of the nebula because of the difficulty of seeing isolated points in the bright portions. One comparison star is the north component of the double star that lies near the center of the nebula; each of the others lies near one of the measured nebulous points. The plate constants were derived from the measures of the comparison stars by a least-squares solution; with these constants the motions of the nebulous points were then referred to the mean of the comparison stars.

The resulting displacements of the nebulous points during the $11^1/_2$ year interval are as follows:

Point	$\Delta\alpha$	$\Delta\delta$	Point	$\Delta\alpha$	$\Delta\delta$
1	+0.22	+0.70	7	−0.22	+0.11
2	+0.43	+1.24	8	+0.32	−1.56
3	−1.24	+2.05	9	+1.78	−1.78
4	−1.67	+1.24	10	+2.05	−0.27
5	−1.12	+1.59	11	+1.78	+0.70
6	−2.16	+1.19	12	+1.02	−0.05

The probable error of a measurement of a comparison star is ±0.23 in right ascension and ±0.25 in declination. The points measured and the observed motions are shown in Plate I, in which the lines represent the displacements that would take place in 500 years if the motion should continue at the rate indicated by the measures. The illustration shows at a glance the random distribution of the motions of the stars and the systematic character of those of the nebulous points. The mean displacements, projected upon radii drawn from the estimated center of the nebula are:

For the comparison stars,	−0.06
For the nebulous points,	+1.54

The large positive value for the nebulous points clearly indicates an outward motion, but it would be premature to suppose that this result necessarily applies to the nebula as a whole. There is some indication of a rotation in the counter-clockwise direction, but this can hardly be regarded as trustworthy.

It may be readily shown that, for a displacement of 2″ in the interval between the two plates, the number representing the velocity in kilometers per second would be about one-fourth the number of light-years in the distance of the nebula from the solar system; for example, if the velocity were 25 km./sec. the distance would be 100 light-years. Thus it is necessary to assume neither an extraordinary distance nor an extraordinary velocity in the nebular particles in order to believe that the observed motions are real.

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[1] *Harvard Observatory Bulletin, Boston,* No. 743, 1921; *Publications of the Astronomical Society of the Pacific, San Francisco,* **33,** 1921 (79–84).

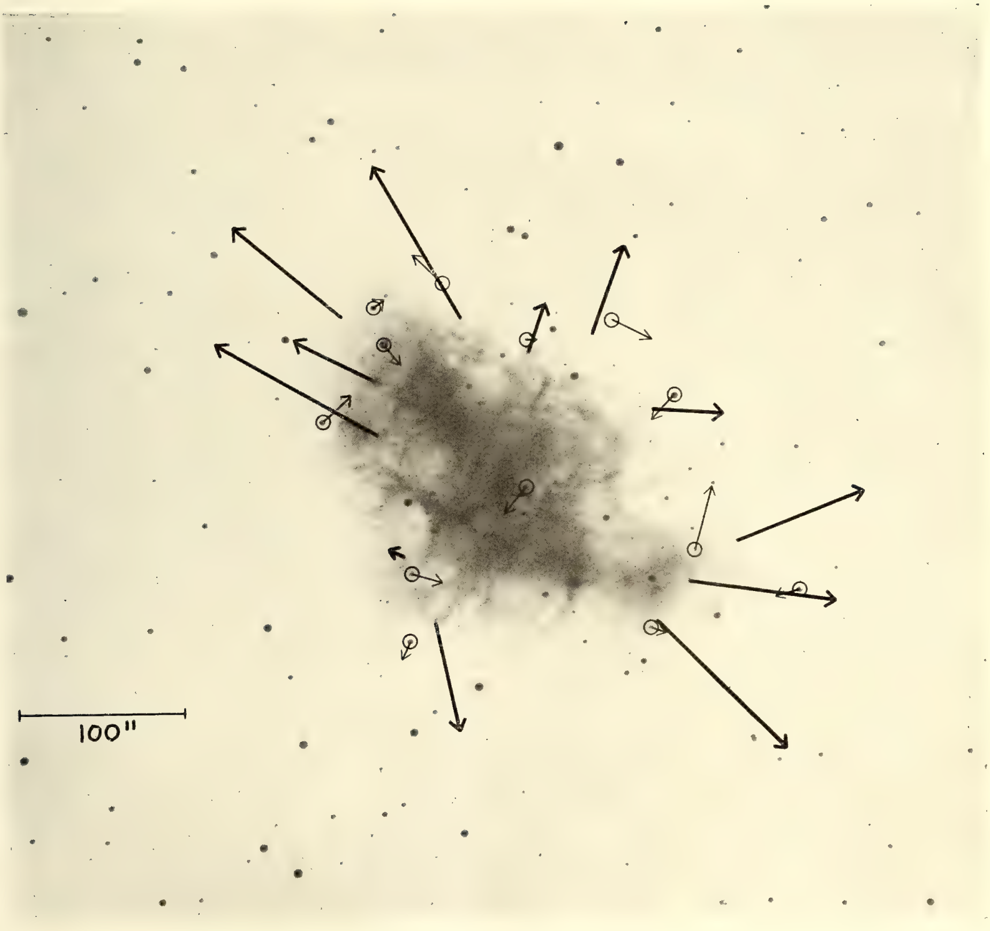

PLATE I

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PROCEEDINGS

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Volume 7 JULY 15, 1921 Number 7

LINKAGE VARIATION AND CHROMOSOME MAPS

By A. H. Sturtevant

Carnegie Institution of Washington

Communicated by T. H. Morgan, March 10, 1921

A recent paper in this journal by Detlefsen[1] is introduced as follows:

> "There is a well intrenched concept of recent genetics that hereditary factors or genes may be given fairly definite loci on chromosome maps and that these maps correspond to or represent, roughly perhaps, the actual conditions in the chromosome. The basis for this attractive and suggestive view is the premise that the distance between two genes is necessarily proportional to the percentage of crossing over which these two genes show, other things being equal. If the distance which gives one per cent of crossovers is used as an arbitrary unit of measurement, then it follows that distances on the chromosome may be calculated in terms of this unit. It has seemed to me for some time that the antecedent in this hypothetical proposition contains a more or less gratuitous assumption. We do not know that the distance which gives 1% (or *n*%) of crossovers is a fixed unit. Stated differently, we do not know how constant the percentage of crossing over may be between two genes to which we give a fixed distance, i.e., our arbitrary unit of measurement may itself prove to be a variable. It may be possible for the distance which gives 1% of crossing over to differ in different females of the same population, or differ between stocks. In order to throw some light on these questions I began a set of experiments in 1916........."

Detlefsen then gives an account of an experiment in which crossover values for the white and miniature loci (in the X-chromosome of Drosophila melanogaster) ranging from 0% to about 33% have been obtained. Crosses between "high" and "low" lines are taken to indicate that a number of genetic factors influence the percentage of crossing over. The paper closes as follows:

> "In view of these considerations it would perhaps be simpler to conclude that linkage is not a function of distance, i.e., crossing over is not necessarily proportional to distance. The distance between two genes may remain fairly constant, but the amount of crossing over depends upon numerous hereditary factors."

One unfamiliar with the literature of the subject would probably infer from Detlefsen's paper that the possibility of inherited linkage variations had not been taken into account by those concerned in constructing chromosome maps. In point of fact, the matter has not only been taken into account, but has been often discussed in the literature, as the following references will show.

In the first paper (1913) in which an attempt was made to construct a chromosome map,[2] the statement occurs (p. 49):

> "Of course there is no knowing whether or not these distances as drawn represent the actual relative spatial distances apart of the factors.... we have no means of knowing that the chromosomes are of uniform strength, and if there are strong or weak places, then that will prevent our diagram from representing actual relative distances..."

Later than this[3] definite evidence for the existence of genes modifying the amount of crossing over was reported, and in the "Mechanism of Mendelian Heredity"[4] the situation was discussed as follows (pp. 67–68):

> "It is not supposed, however, that the per cent of crossing over represents precisely the distance between the factors, for it may be that crossing over is more likely to take place in one region than in another. In that case the distances between factors in this region calculated from the amount of crossing over between them, would be relatively greater than the actual distance.... Sturtevant has found definite factors which alter the amount of crossing over in the chromosomes, and these factors actually do affect the amount of crossing over differently in the different regions......however, ...the *order* of the factors remains unchanged."

The three papers mentioned above were all published before Detlefsen says he began his experiments, as was also Bridges,[5] demonstration that frequency of crossing over in the second chromosome changes with the age of the female. Since that time the question has been discussed again and again, and several cases of inherited linkage variations have been thoroughly analyzed.[6] The conclusions to be drawn from such evidence have been well stated by Morgan,[7] in a chapter devoted entirely to variationsin linkage (p. 125):

> "It is to be understood, then, that when we substitute the idea of distance for crossing over values the term is not used in an absolute sense, but in a relative sense, and that it depends always on the conditions of the experiment. That the genes do stand at definite levels in the chromosome, and that in this sense they are definitely spaced, seems reasonable in the light of all the evidence bearing on this point; but even if they are so spaced that crossing over is a function of their distance from each other in the series, any influence that determines how often interchange between homologous pairs will take place would give the appearance that the actual distances themselves have changed."

As will be evident from the quotations given, the chromosome maps are intended to show the actual sequence of the loci, and the relative amounts of crossing over between them. The intervals between adjacent loci are not to be taken as necessarily proportional to the actual spatial distances between them—though that distance is evidently one of the elements concerned. When the amount of crossing over changes from any cause it is evident that we are dealing with a new system, and the intervals of the "normal" map will no longer be applicable (though in all cases so far investigated the *sequence* of loci shown in the "normal" map is unchanged). The "normal" map itself is based on the average results obtained under conditions in which no recognized disturbing factors (genetic or environmental) were known to be present. This map represents the results to be expected from untried crosses, and has shown itself to be invaluable in everyday work. Detlefsen's own account is in agreement with this, for he says that his stock of white miniature has been used in class work and has always given the value to be expected from the published maps. When values that do not agree with the "normal" map are found, analysis has always shown some disturbing factor to be present; and in all cases where the point has been investigated it has been found that maps based on data uniform with respect to this disturbing factor are entirely self-consistent.

In view of these considerations it is clear that Detlefsen has misunderstood the significance of the published maps. The conception that he has attacked is one that has not been held or urged by those who have constructed chromosome maps.

[1] Detlefsen, J. A., These PROCEEDINGS, **6**, 1920 (663–670).

[2] Sturtevant, A H., *J. Exper. Zoöl.*, **14**, 1913 (43–59).

[3] Sturtevant, A. H., *Zs. ind. Abst. Vererb. Lehre*, **13**, 1015 (234–287).

[4] Morgan, T. H., A. H. Sturtevant, H. J. Muller, and C. B. Bridges, New York, 1915, 262 pp.

[5] Bridges, C. B., *J. Exper. Zoöl.*, **19**, 1915 (1–21).

[6] Sturtevant, A. H., These PROCEEDINGS, **3**, 1917 (555–558), and *Carnegie Inst. Washington Publ.*, No. 278, 1919 (305–341); Muller, H. J., These PROCEEDINGS, **3**, 1917 (619–626), and *Genetics*, **3**, 1918 (422–499); etc.

[7] Morgan, T. H., *The physical basis of heredity*, Philadelphia, 1919, 305 pp.

ORIGIN AND HISTORY OF THE BEAR FAMILY IN THE WESTERN HEMISPHERE, WITH PARTICULAR REFERENCE TO THE RELATION OF THIS QUESTION TO PROBLEMS OF GEOGRAPHICAL HISTORY

BY JOHN C. MERRIAM

CARNEGIE INSTITUTION OF WASHINGTON

Read before the Academy, April 26, 1921

Although bears of many varieties are widely distributed in America and have been present in large numbers for a very long period measured in terms of years, the history of this group shows that as we go back in the geological record no evidence of representatives of the bear type are present in formations of the third geological period preceding the present in America, while they are known in considerable numbers in the rocks of this age found in the Old World. There is, therefore, good reason to believe that the bear group is derived from the other side of the earth and that the ancestors of the present American bears migrated to this continent at a time geologically not far removed from the present.

The bears of the world may be divided into two large groups, one sometimes known as the true bears, including the brown and grizzly types now widely distributed over the whole northern hemisphere. The second group is represented by the spectacle bear and is limited to the Andean region of South America.

In the geological period known as the Pleistocene, immediately preceding the present, true bears like the present black bear type were associated in North America with a group of large bears known as Arctotheres, closely related to the present spectacle bear of South America. In the same period South America was inhabited by numerous Arctotheres but contained no representatives of the true bears.

In the rocks of the Pliocene, or second geological period preceding the present, until recently no representatives of the bear group have been known in either North or South America. In the Pliocene deposits of Europe and Asia there are, however, remains of creatures closely related to the true bears and with these a second group originally known as the hyena bears, or the Hyaenarctos type, closely related in many characters to the Arctotheres, and through them related to the modern South American spectacle bears.

Assuming that the American bears are descendants of the Old World types, it is difficult to escape the conclusion that wide land connections existed between North America and Asia, and between North and South America, at such times as to permit the migration suggested by the present distribution. As the Arctotheres are represented in the Pleistocene of South America without accompanying true bears, it is logical to assume that the Arctothere group was the first to migrate to America and that it passed through North America reaching the South American region before the true bears had spread over North America. Investigations within the last few years have shown that in the Pleistocene deposits of North America both Arctotheres and true bears are present together down to the earliest strata in which remains of bears have been discovered. There has, therefore, been reason to assume that remains of Arctothere-like bears would be found in the Pliocene of America without associated remains of the true bear type. The point of the present paper is to call attention to the fact that recent carefully conducted investigations in the Pliocene deposits in several parts of North America have brought to light remains of bear-like forms which are intermediate between the typical hyena bears of the old world Pliocene and the Arctotheres of the American Pleistocene. Such a form is represented by a specimen found in the Pliocene of Oregon. This animal corresponds very closely to the most specialized of the hyena bears of India and also approaches the Arctotheres in its structure.

The evidence now available indicates that creatures of the hyena bear type came by way of broad land connections from Asia to North America in Pliocene time; that these creatures represent the type of hyena bear most nearly approaching the Arctotheres and were widely distributed in North America. There is reason to believe that from this group the Arctotheres may have developed within the American region, and that the Arctotheres by way of a wide land bridge came to people South America.

The present spectacle bears of South America seem then to represent the last remnant of a group which originated in the Old World, was once widely distributed over the world, and included the largest of all known bears.

AN UNIDENTIFIED BASE AMONG THE HYDROLYTIC PRODUCTS OF GELATIN

Donald D. Van Slyke and Alma Hiller

Hospital of the Rockefeller Institute for Medical Research, New York, N. Y.

Communicated June 25, 1921

Of the known amino acids yielded by acid hydrolysis of proteins the work of various authors[1] has indicated that four, viz., histidine, arginine, lysine, and cystine, are distinguished from the others by the relative insolubility of their phosphotungstates in acid solution. On the basis of this fact Van Slyke[1] devised a method for separating these four amino acids as phosphotungstates, and determining them by utilizing certain characteristics of their chemical structure. The non-amino nitrogen of this group of amino acids is entirely in the histidine and arginine. The arginine was determined directly, and the histidine was estimated on the assumption that all the remaining non-amino nitrogen was in histidine.

This assumption we have tested by comparing the histidine content of a number of proteins as determined in the above manner with the values determined by Koessler and Hanke's direct colorimetric method.[2] In casein, edestin, and fibrin, the results by the two methods agree. But in gelatin the calculation based on the non-amino nitrogen indicates 6.1 per cent of the total protein nitrogen in the form of histidine, while the colorimetric method shows only 1.8 per cent. There is evidently among the products of gelatin hydrolyzed by hydrochloric acid a substance, or substances, hitherto unrecognized, precipitated with phosphotungstic acid under the conditions ordinarily utilized to precipitate the hexone bases.

In attempting to isolate the substance we have precipitated it by means of phosphotungstic acid with the other bases, have redissolved the precipitate and freed it from phosphotungstic acid. The histidine and arginine were removed by precipitation with silver sulfate and barium hydroxide, and the lysine as picrate. The residual solution contained an amount of non-amino nitrogen corresponding approximately to that determined by Van Slyke's method in excess of the arginine and histidine non-amino nitrogen.

Attempts to crystallize the free base or its derivatives have been successful only with the phosphotungstate. Recrystallization of the phosphotungstate yields a product in which the ratio, *Total N : Amino N* = 2 : 1. The free base prepd. from the recrystallized phosphotungstate is hygroscopic, and decomposes slowly when dried at 100°. It does not appear to be a peptide, for the ratio of amino to total nitrogen is not increased by boiling 40 hours with 20 per cent hydrochloric acid, nor by heating 8 hours in a bomb tube at 125° with 25 per cent sulfuric acid. We are engaged in the preparation of larger amounts of the substance in the hope of determining its structure.

[1] Literature quoted by D. D. Van Slyke, *J. Biol. Chem.*, **10,** 1911 (15).

[2] Koessler, K. K., and Hanke, M. T., *J. Biol. Chem.*, **39,** 1919 (497).

GENETICAL AND CYTOLOGICAL PROOF OF NON-DISJUNCTION OF THE FOURTH CHROMOSOME OF DROSOPHILA. MELANOGASTER[1]

BY CALVIN B. BRIDGES

CARNEGIE INSTITUTION OF WASHINGTON

Communicated by T. H. Morgan, March 18, 1921

A mutant type of *D. melanogaster* known as "Diminished" gave genetical results which proved the involvement of a "deficiency,"[2] i.e., a multilocal loss of genes, in the chromosome corresponding to the "fourth linkage group"[3,4] (see section I). Further exceptions to normal inheritance showed that non-disjunction[5] of this chromosome had occurred giving rise to (Diminished) individuals lacking one member of the fourth-chromosome pair (see section II). The deficiency in this case consisted therefore in the loss of an entire chromosome. The haploid nature of Diminished was then proved cytologically: it was found that in the cells of Diminished individuals only one small round chromosome was present instead of a pair (section III). This finding demonstrates the correctness of the view that the carrier of the genes of the fourth linkage group is the small round chromosome. A positive direct proof is provided that a particular autosome is the carrier of the genes of particular non-sex-linked Mendelian characters.

I. The features that in the aggregate prove that a deficiency is involved are:

(1) A definite complex of mutant characters: Diminished differs somatically from the wild-type in many characters, among which the principal are: smaller size throughout; bristles relatively shorter and more slender; body-color paler; thorax-pattern darker; eyes larger and roughish, somewhat denuded of hairs; aristae reduced or absent; wings slightly spread, blunter, and less clear in texture.

(2) Low productivity.

(3) Frequent stirility.

(4) Dominance.

(5) Heavy mortality: Outcrosses of "heterozygous" Diminished to wild-type gave an extremely variable proportion of Diminished; the bulk of the cultures ranged from 20 to 50 per cent, with a mean value of 34.5 (1772 Dm: 3366 wild-type). Since the Diminished offspring were expected to be as numerous as the wild-type, the mortality of the Diminished is very high, viz. 47%.

(6) Delayed hatching: Diminished adults begin to emerge from the pupa cases four or more days later than wild-type sibs. The delay is apparently due to a slow metamorphosis.

(7) Lethal when "homozygous:" When "heterozygous" Diminished was inbred there were 139 Diminished to 100 wild-type (1079 : 775). The expectation for such inbreeding, on the basis that "homozygous" Diminished is lethal and that mortality of the "heterozygous" Diminished is the usual 47%, is 106 : 100, which is somewhat under the observed ratio. The mortality of Diminished is extremely variable, and a mortality of 31% would give the observed ratio. Further evidence in favor of the lethal view is the fact that continued inbreeding did not increase the percentage of Diminished; also no Diminished individual was found that gave all Diminished offspring upon outcrossing.

(8) Inheritance as member of the fourth chromosome:

(*a*) The Diminished mutant is not due to a change in the first, second, or third chromosomes: Reciprocal crosses gave like results, and outcrosses of Diminished male to wild female gave Diminished offspring among the sons, so the Diminished is known to be neither cytoplasmic nor sex-linked in inheritance. Male back-cross tests with such well-known second-chromosome mutants as Star and black, and third-chromosome mutants, such as Dichaete, Hairless, pink and ebony, showed free assortment of these mutants with Diminished. Since there are only four chromosome pairs in *D. melanogaster* the above evidence proves that Diminished is caused by a change in the remaining or "fourth" chromosome.

(*b*) That Diminished is due to a fourth-chromosome change might be proved by means of male back-cross tests with the known fourth-chromosome recessive mutants bent and eyeless. But when crosses of Diminished to bent and to eyeless were made in order to secure the necessary heterozygotes, it was found that all the F_1 flies that were Diminished were at the same time bent or eyeless, respectively. This fact made the intended back-cross tests both impossible and unnecessary. Crosses of bent, or of eyeless, to wild and to other mutants give only not-bent or not-eyeless progeny in F_1, while, on the other hand, crosses of Diminished to a great variety of recessives in the first, second, and third chromosomes have never given F_1 Diminished that showed any of those recessives. The fact that the Diminished allows "pseudo-dominance" of two dissimilar non-allelomorphic fourth-chromosome recessives proves that Diminished is due to a fourth-chromosome change, and one of a definite known class—that of "deficiencies."

(9) Pseudo-dominance of recessives in the affected section: Pseudo-dominance, such as found for the recessives bent and eyeless with Diminished, had already been met with in such cases as forked with Bar-deficiency,[1] vermilion with vermilion-deficiency,[6] and facet, abnormal and the white allelomorphs with Notch-deficiencies.[7] The explanation in these cases is that the section of chromosome from forked to bar inclusive, the section containing the vermilion locus, and the section from white to facet and abnormal inclusive, had been lost, or transformed in such a way as to have lost the normal action of its genetical materials. All previously enumerated features of Diminished are explainable by deficiency for a section of the fourth chromosome between and inclusive of the loci for bent and eyeless. A diminished individual is virtually haploid for this section.

The extent of a deficient region is measured in two ways: by observing to how many neighboring mapped loci pseudo-dominance extends, and by observing the amount by which the map-distance between adjacent not-included loci is decreased in the presence of the deficiency, there being no crossing over within the deficient region. The Diminished deficiency included the only two known fourth-chromosome mutants and thus made the second method impossible in this case.

(10) Exaggeration of mutants of affected section: The characters bent and eyeless in flies that are at the same time Diminished are more exrteme departures from the wild-type than are the simple bent and eyeless mutants. For example, the legs of the Diminished-bent compounds were much shorter and all the tarsi were like those of the hind legs of the most extreme normal bents; i.e., they were fused into hairy lumps instead of being chains of five slender joints. The eyes of the Diminished-eyeless compounds that hatched had about the same range as normal eyeless has; but among the dead pupae were found individuals in which the head was reduced to a mere proboscis on the end of the neck.

(11) Exaggeration of mortality of mutants of affected section: Crosses of Diminished to eyeless produced only 35 Diminished-eyeless (all males) to 2726 wild-type sibs, where equality is expected. The mortality of eye-

less in presence of Diminished is thus 98.7%. The mortality of simple eyeless is usually not greater than 5%, and the mortality of the simple Diminished is, as seen, about 47%. The mortality of Diminished-bent was even higher, viz. 99.5%, since only 9 Diminished-bents (males) hatched for 1727 wild-type sibs.

Since mortality changes must necessarily be only consequences of somatic changes, the exaggeration of the mortality of the Diminished compounds is another evidence of exaggeration of somatic changes—probably of the internal anatomy in addition to the observed superficial changes.

The mortality of Diminished, and of the Diminished compounds, is mainly due to somatic changes in the pupa stage: (*a*) If the mortality of Diminished individuals were due to changes in the larvae, then mass-cultures, because of the greater competition among the too numerous larvae, should show a lower proportion of Diminished than do sister pair cultures, but this was found not to be the case. (*b*) Examination of the culture bottles of the crosses of Diminished by eyeless showed that the number of Diminished-eyeless individuals that pupated but perished subsequently is approximately the same as the number of not-diminished flies that hatch.

(12) Exaggeration of certain mutants outside the affected region: Diminished exaggerates the third-chromosome dominant Hairless, and to a slight extent certain other mutants.

(13) Recurrent nature: Diminished has arisen independently at various times (see section II (1) below).

II. All the above features of Diminished, including exaggeration (10, 11, 12), are explainable by the theory of deficiency, and for nearly all there have been parallels in previously known cases of deficiency. Two features of Diminished appeared that made a sectional deficiency hypothesis inadequate.

(1) High frequency of re-occurrence: Diminished has been recorded some twenty-five times within the space of a year, which is too high a frequency to be explained without extreme difficulty on the sectional deficiency view. Notch-deficiencies have been recorded some 18 times, but this was in a period extending over eight years. Furthermore, there is evidence that the different Notches are not re-occurrences of the same deficiency, but show character differences and involve sections that are different in extent, though all possessing a common section in the neighborhood of facet. The different Diminished strains, on the oteher hand, are apparently identical. Most of the tests of this paper were carried out in parallel with strains No. 3, 4, 8, 15, and 3x.

(2) Exceptions to normal inheritance: In a cross of Diminished to bent there appeared a single Diminished male that was not bent. This exceptional male was tested and was found not to carry bent, although

his father had been homozygous bent. Furthermore, in a cross of Diminished to eyeless there appeared a single not-Diminished eyeless female. Genetical tests showed that this exceptional female was homozygous for eyeless, although her father had lacked eyeless entirely.

These exceptions to normal inheritance invited a comparison with the well-known exceptions to sex-linked inheritance which are due to non-disjunction of the sex chromosomes;[4] for it is evident that non-disjunction of the fourth chromosome would produce such exceptions, and also that it would give rise to the whole series of effects that are characteristic of deficiency. Diminished individuals are deficient not simply for a section of the fourth chromosome including the loci for bent and eyeless, but are deficient for a whole fourth chromosome; i.e., they are haploid for the fourth chromosome. A Diminished strain would arise from the occurrence of "primary" non-disjunction in either sex, whenever at maturation the pair of fourth chromosomes fails to disjoin and both members pass to the same pole, leaving the other pole without a fourth chromosome. Such nullo-IV eggs or sperm in union with normal haplo-IV sperm or eggs would give rise to individuals with only a single fourth chromosome. Judging by the number of independent strains of Diminished and by the two controlled exceptions, the frequency of primary non-disjunction of the fourth chromosome is of the same order as for the first.

If primary non-disjunction gave rise to nullo-IV gametes and thus to haplo-IV strains, the same process may also give rise conversely to diplo-IV gametes and thus to triplo-IV zygotes. No such individuals have as yet been indentified.[8] Aside from possible direct somatic effects of triploidy or tetraploidy for the IV chromosome, the presence of an extra chromosome would be detectable by duplication-phenomena,[9] just as the haplo-IV type was detectable by deficiency phenomena, and could be proved cytologically. The eyeless exception referred to above was produced by a diplo-IV egg, so that the process is well substantiated even here.

III. Cytological examination of the oögonial cells of Diminished individuals showed, as had been predicted, that only a single one of the small spherical chromosomes was present, while each of the other chromosomes was present in the usual paired condition. Four independent strains of Diminished, including the one that arose in the cross, were thus proved to be haploid for the fourth chromosome (fig.1).

IV. The haploid nature of the Diminished individuals made possible a back-cross determination of the bent eyeless crossover value without the necessity of a double recessive. The situation in the fourth chromosome is now comparable with that in the first, with haploidy for the fourth chromosome giving the Diminished type and pseudo-dominance of fourth-chromosome recessives, while haploidy for the first gives the male type and pseudo-dominance of sex-linked recessives. Females heterozygous for bent and for eyeless were crossed to Diminished males, and the offspring furnished the equivalent of 17 crossovers in a total of 1987, or 0.86 ± .14 per cent of crossing over.

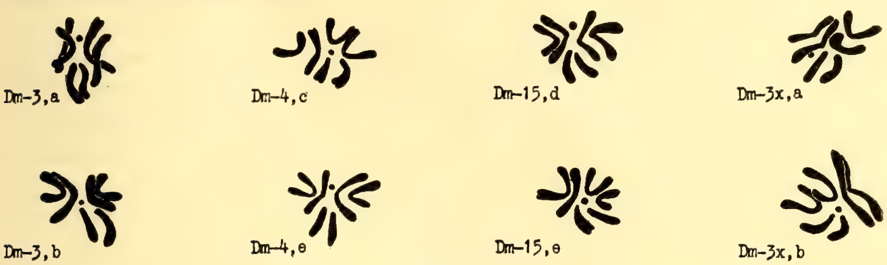

FIG. 1

An oögonial metaphase plate from each of eight "diminished" individuals, two from each of four strains, all figures showing a single round chromosome instead of the usual pair.

V. The cytological finding in the case of the haplo-fourths has put the theory of deficiency on a sound basis; for, each of the features met with in the cases described as deficiency has been met with in the case of the haplo-fourths. Thus, it has been shown, not simply that these characteristics might be produced by loss, but that in this specific case they were actually produced by loss.

VI. Both the character complex and the exaggeration effects of Diminished are due to a change in balance of the genes. Each character of the wild-type is the result of a balance between many genes that tend to modify the character in various directions. These "plus and minus modifiers" are apparently distributed at random in the chromosomes, so that if a particular short section of chromosome is removed, as in deficiencies, usually more plus than minus modifiers (or vice versa) of some character will be removed. In that case more minus than plus modifiers will remain in action, and a new character will appear, differing from the old in a minus direction.

VII. The genetic and cytological evidence in the case of Diminished has provided a positive identification of a particular chromosome—the small round one—as the carrier of the genes of particular Mendelian characters—bent and eyeless. Non-disjunction of the sex-chromosomes has previously furnished a similar identification of the rod-like chromosome as the carrier of the genes of certain sex-linked characters. Further work on non-disjunction of the X-chromosome has extended the identification to include so many sex-linked characters without exception that it may be accepted as general. Since the behavior of non-sex-linked characters had been shown to be the same as that of the sex-linked characters in a large range of particulars—in all points except those connected with the fact that the male is virtually haploid for this chromosome—the demonstration that the sex-linked genes are carried by the X-chromosome left no plausible ground for believing that that proof did not hold for the ordinary characters and the ordinary chromosomes. However, the present case offers a direct and positive identification between particular ordinary Mendelian characters and a particular autosome, so that we need no longer depend upon an extension to ordinary characters of a proof that was complete only for sex-linked characters. Taken in connection with the immense body of general proof for the chromosome theory, and with the many established parallels between hereditary behavior and chromosome behavior, these direct proofs in the case of both sex-linked and autosomal characters may be regarded as decisive.

[1] Paper read before the Genetics Seminar of the University of California, Dec., 1920.

[2] Bridges, C. B., *Genetics*, **2**, Sept., 1917 (445–465).

[3] Muller, H. J., *Jour. Exp. Zool.*, **17**, Oct., 1914 (325–537).

[4] Hoge, M. A., *Am. Nat.*, **49**, 1915 (47).

[5] Bridges, C. B., *Genetics*, **1**, Jan.–Mar., 1916 (1–52, 107–163).

[6] Bridges, C. B., *J. Gen. Physiol.*, **1**, July, 1919 (645–656).

[7] Mohr, O. L., *Genetics*, **4**, May, 1919 (275–282).

[8] Dr. Little has recently reported such an identification (*Science*, **53**, Feb. 18, 1921 (167)); but as I have pointed out in a note to *Science* (**53**, April 1, 1921 (308)) the facts as stated are more in conformity with the assumption of a "weak" allelomorph of eyeless, or of a linked dominant "minus" modifier of eyeless. (Since the above was written, triplo-fourth individuals have been identified among the offspring of triploid females (*Science*, **54**, Sept. 16, 1921 (252)).

[9] Bridges, C. B., *Proc. Am. Soc. Zool.*, 1918.

NOTE ON THE ECONOMIC CONVERSION FACTORS OF ENERGY[1]

BY ALFRED J. LOTKA

BROOKLYN, NEW YORK

Communicated by R. Pearl, June 20, 1921

That there is some relation between physical energy and economic value has been more or less clearly recognized by several authors. So G. Helm, for example, conceives of an "economic energy," whose capacity factor is quantity (of commodity), and whose intensity factor is measured by the price paid per unit quantity. A certain justification for this standpoint may be found in the fact that on the one hand goods having economic value can be exchanged for, and in this sense converted into energy; while on the other hand the movement of commodities in an economic system, that is to say, the transfer of property by sale and purchase, is determined by differences in economic value in much the same way as the transfer of energy from one portion of a physical system to another is determined by differences in the intensity factor of the energy concerned in the change.

From Helm's[2] standpoint, money would represent the most fluid form of economic energy, convertible freely into any other form of economic energy. Attention to this rôle of money is also drawn by Ostwald, who, however, expressly makes the reservation that money, although bearing a certain resemblance to energy, is not essentially identical therewith.[3]

Nevertheless, as we all know, money is convertible, by purchase upon the market, into various forms of energy, at rates somewhat variable, yet in some manner determinate.

The question therefore arises, what is it that determines the ratio of conversion of various forms of energy by exchange upon the market?

The physicist is familiar with two kinds of conversion factors of energy.

The first kind of factor is that which enters into the analytical expression of the first law of thermodynamics, the expression of the constancy in ratio of the amounts of several forms of energy replacing each other in physical transformations.

The second kind of factor relates to that fraction of energy transformed, which can be recovered in a selected form, and expresses the "efficiency" of the transformation or of the transformer taken in view.

It is one of the central data of thermodynamics that the equivalence factor is always independent of the mechanism by which the transformation is effected, while the efficiency factor is thus independent in the ideal case of a reversible transformation

It is this independence which makes the two laws of thermodynamics so fertile as tools for drawing conclusions regarding the course of physical events. Anyone who has ever sought to show a perpetual motion inventor just why his machine cannot work will appreciate the economy of thought and language which is secured by direct application of these principles.

Now in the transformation of energy by economic exchange upon the market we are dealing with a third type of conversion factor. The physical relations here involved are so complex that we are apt to overlook altogether that they *are* physical. Still less do we ordinarily recognize their precise character.

A simple example may help to clarify the view: the case of the automatic vending machine, the penny-in-the-slot chocolate dispenser, for instance.

The salient facts here are:

1. A definite amount of money brings in exchange a definite amount of commodity (and of energy).

2. The physical process is a typical case of "trigger action," in which the ratio of energy set free to energy applied is subject to no restricting general law whatever (e.g., a touch of the finger may set off tons of dynamite).

3. In contrast with the case of thermodynamic conversion factors, the

proportionality factor is here determined strictly by the particular mechanism employed.

Reflection shows that all transformations of money or of economic assets of any kind into energy by exchange upon the market are of this character. It is always a case of trigger action. Somewhere there is a store of available energy, which can be tapped with an expenditure of greater or less effort. The payment of the price sets in motion the requisite machinery for the release of that energy (or for its transfer of ownership, the release being delayed at the discretion of the buyer).

In view of the entire absence of any general law regulating the ratio of energy released to energy applied in such case of trigger action, we may ask the question, how does it come about that economic conversion factors, economic ratios-in-exchange of different forms of energy, display any regularity whatever?

We have been accustomed, in thermodynamics, to disregard mechanism. In dealing with energy conversions by trigger action we must make a complete change of attitude. Here everything depends on mechanism. If we find any degree of regularity in the conversion factors, this must be due to regularities in the mechanism, i.e., in the human organism, and its social aggregations; and we cannot hope to make any progress in our understanding of the physical principles involved except by taking due account of the mechanism.

In view of the extreme complexity of the systems in which economic phenomena have their course, it may appear hopeless to attempt analysis. But reflection shows that the complexity is in the details; the broad underlying features resolve themselves into comparatively simple lines.

Our vital interest in physical energy arises out of the following facts:

1. The body of a living organism is *not* in equilibrium with its surroundings. In other words, the living body (and also the dead body, for some time after death, as popularly conceived), in its natural environment, is a source of available energy.

2. Though not in equilibrium with its surroundings, the living body is in an approximately steady state; furthermore, it is one of the essential conditions for the continuance of life that this approximately steady state be maintained. Extreme departures therefrom result in permanent displacement from the steady state, and, ultimately, in the dissipation of the available energy presented in the substance of the organism.

3. In the course of events a certain amount of available energy, varying according to the nature of the organism, is unavoidably dissipated, per unit of time.

It follows that every organism, in order to maintain the steady state necessary for its survival, must be provided with devices for capturing available energy.

Furthermore, in the competition which takes place among organisms, the advantage must go to those whose energy-capturing devices are most effective in directing available energy into such channels as are favorable to the preservation of the species.

In man, as an organism of the animal type, the mechanism for capturing available energy comprises three elements:

1. Sense Organs, or Receptors, whose function is to establish a certain rather close correlation between the state of the environment and that of the individual. Their function, in fact, is to *depict* the external world in the organism, to apprise him of the state of his environment.

2. Organs of Operation, or Effectors, such as hands, feet, etc., by means of which the individual reacts physically upon his environment so as to modify it, or to modify his relation to it (as, for instance, by locomotion).

3. Organs and Faculties of Adjustment, or Adjustors, whereby the action of the Effectors is adjusted in accordance with the indications furnished by the sense organs (receptors), and with the needs of the organism.

Among these adjustor faculties one which figures prominently and plays an important rôle is what we may call the *sense of values*, that faculty which we exercise when we are confronted with two or more alternative courses of action, from which one must be selected. In such a situation we choose the course which appears to us, subjectively, the more *desirable*, and we do not ordinarily give any thought to the question as to what may be, objectively, the significance of "desirability," any more than we do ordinarily concern ourselves, in viewing a landscape, as to the particular wave-length, the objective characteristic, of the light which to our subjective judgment appears "green."

And yet, for the interest of the species, it is evidently far from indifferent what may be the objective characteristics of those things which to us appear, subjectively, desirable. Upon the proper adjustment of the sense of values, of the "tastes," to certain objective realities depends the welfare of the species. A nation of drunkards, for example, is not destined to figure among the winners in the struggle of evolution.

If, then, the physicist has been interested in discovering the objective significance of the indications of our sense organs, if he has thought it within his province to investigate the relation between color and wavelength, between musical pitch and frequency, then it is equally a problem for the physicist to enquire into the physical basis of economic value.

The ground has been prepared for this enquiry by the mathematical economists in their hedonistic calculus. A person having at his disposal a given sum of money, and being confronted with the necessity of choosing between different ways of spending it, will seek so to apportion his expenditures as to procure for himself a maximum of pleasure. So, for example, if the choice lies between two commodities C_1 and C_2 (which may, in particular, be certain forms of energy) he will purchase quantities q_1 and q_2, respectively, such that, if his total pleasure derived from the purchase is Ω, then

$$\frac{\partial \Omega}{\partial q_1} dq_1 + \frac{\partial \Omega}{\partial q_2} dq_2 = 0 \tag{1}$$

where dq_1, dq_2 are small, arbitrary increments in the quantities of the commodities C_1, C_2 purchased. If p_1, p_2 are the prices upon the open market, say, per unit quantity, of commodities C_1, C_2, respectively, then evidently the corresponding increment in expenditure is $p_1 dq_1 + p_2 dq_2$. But the sum of money to be spent is fixed. We must have, therefore,

$$p_1 dq_1 + p_2 dq_2 = 0 \tag{2}$$

It follows, then, from (1) and (2), that the individual will so apportion his purchases, that

$$\frac{\partial \Omega}{\partial q_1} \Big/ \frac{\partial \Omega}{\partial q_2} = \frac{p_1}{p_2}. \tag{3}$$

If the commodities C_1, C_2 are "necessities of life," whose influence upon the life of the species is to increase r, the proportional rate of increase of the species, then, evidently, the interests of the species demand that the purchases be so apportioned as to make r a maximum, and thus, by similar reasoning, to make

$$\frac{\partial r}{\partial q_1} \Big/ \frac{\partial r}{\partial q_2} = \frac{p_1}{p_2}. \tag{4}$$

If we regard the sense of values as a device for attaining, as nearly as may be, the adjustment (4), then we see that a perfect sense of values would make

$$\frac{\partial \Omega}{\partial q_1} \Big/ \frac{\partial \Omega}{\partial q_2} = \frac{\partial r}{\partial q_1} \Big/ \frac{\partial r}{\partial q_2}. \tag{5}$$

So that, in a community of organisms endowed with a sense of values thus perfectly adjusted, the marginal utilities $\frac{\partial \Omega}{\partial q}$ (and hence the prices on an open market) would be proportional to the partial derivatives $\frac{\partial r}{\partial q}$. In practise this proportionality will not be actually reached; but it represents the limiting state towards which a race must tend to evolve. For, a race having tastes radically at variance with the exigencies of race propagation (measured by the rate of increase r), could not long survive, as pointed out years ago by Herbert Spencer.[4]

It is this tendency to approximate (perhaps somewhat remotely) to the adjustment (5), which gives rise to fairly definite economic interconversion factors of different forms of energy, as purchased upon the market, and applied to specific uses. There is no need to invoke a specific "economic energy;" on the contrary, our analysis, in which the quantities $\frac{\partial r}{\partial q}$ play a significant rôle, lends little support to Helm's view regarding the physical dimensions of economic value. Such value, or, to use more exact terms, "marginal utility," does not, in the analysis here presented, appear as the intensity factor of an energy.

[1] Papers from the Department of Biometry and Vital Statistics, School of Hygiene and Public Health, Johns Hopkins University, No. 40.

[2] Helm, G., *Die Lehre von der Energie*, Leipsic, 1887, pp. 72 et seq.

[3] Ostwald, W., *Energetische Grundlagen der Kulturwissenschaften*, Leipsic, 1909, p. 155. *Die Philosophie der Werte*, Leipsic, 1913, pp. 260, 314–317, 326, 328. Among other writers who touch on the subject of the relation of economic value and price to energy are: Budde, *Energie und Recht*, Leipsic, 1902, p. 56; Winiarski, "Essay sur la Mécanique Sociale," *Revue Philosophique*, 1900, vol. 49. p. 113. See also J. Davidson, *Qu. J. Economics*, Aug., 1919, p. 717.

[4] And for others whose pains and pleasures he makes his own.

THE BEHAVIOR OF HOMOLOGOUS CHROMOSOMES IN A TRIPLOID CANNA

By John Belling

Station for Experimental Evolution, Cold Spring Harbor

Communicated by C. B. Davenport, June 12, 1921

In a diploid flowering plant we may regard the chromosome group as consisting of a number of sets of chromosomes, two in each set. When, there are differences of size in the group, as has been shown in Yucca,[4] Crepis,[6] Morus,[5] Datura, and two dozen or more other species,[8] the two chromosomes of each set are of the same size; they form bivalents distinguishable by their sizes at the first maturation divisions of the pollen-mother-cells;[1] and they can consequently be seen to replace one another in the haploid groups of chromosomes of the microspores and megaspores. Evidence from breeding, especially in cases of non-disjunction, also shows conclusively that the chromosomes of one pair are nearly equivalent or homologous. We may then define a diploid plant, in the strict sense, as one having a chromosome group formed of pairs of homologous chromosomes. A triploid plant possibly comes either from the cross of a tetraploid and a diploid, or from the union of a normal germ-cell with a pollen-grain or egg-cell in which one of the cytoplasmic divisions of maturation has been omitted. (I have found one cytoplasmic division commonly omitted, especially after cold, in the pollen of Stizolobium, Datura, etc.) The differences in size in the chromosomes of a triploid mulberry,[7,5] and of a triploid Datura, show that the sets consist of three similar chromosomes each. The breeding results with plants in which one set consists of three chromosomes, while the other sets have only two, show that the three are homologous.[2] Hence we may define a triploid plant, in the strict sense, as one having a chromosome group made up of a number of sets of three homologous chromosomes.

In diploid plants there is an attraction of some kind between the two homologous chromosomes which form a bivalent or dyad at the first division. We may expect an attraction also between the three homologous chromosomes in the corresponding triploid plants, leading to the

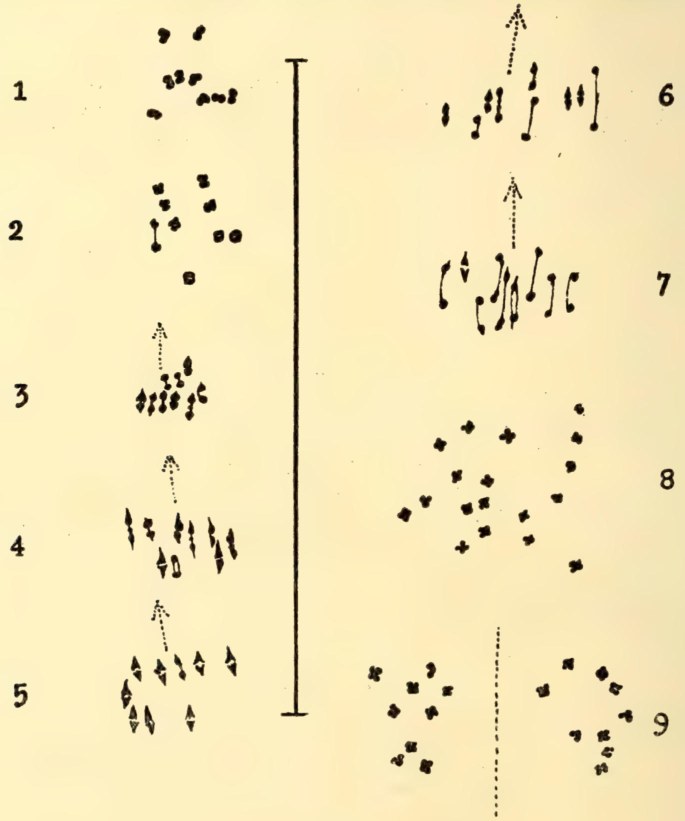

FIG 1

The drawings were made at stage height, with the Abbe camera. A Leitz 2 mm. apochromatic objective of 1.3 aperture, and compensating ocular 6 were used. An achromatic condenser of 0.9 aperture was focussed on the object so as to give critical illumination. The pollen-mother-cells were fixed and stained with iron-aceto-carmine (1).

The assortment of the chromosomes in the pollen-mother-cells of a diploid Canna clone. This was obtained under the name "Madame Crozy." (1) Prophase of the first division, showing 9 bivalents. (2) Metaphase of the first division, showing 9 dyads, one already divided. (3), (4), (5), (6), and (7) are stages of early anaphase, showing division of the 9 dyads into 9 + 9 chromosomes. (The arrows indicate the situation of the polar axis.) (8) After the first division, showing 18 chromosomes. (9) The two groups in the metaphase of second division, separated by a cell-wall.

formation of triads. Thus Wilson[9] observed that three small chromosomes in Metapodius were grouped into a triad in the first division, and separated so that two went to one pole and one to the other. Osawa[5] found in triploid mulberries that some of the single chromosomes might be attached to the dyads in the first division of the pollen-mother-cells. In one triploid Canna (and in a triploid Datura), however, I found that all the chromosomes regularly form triads in the prophase and metaphase of the first division, and pass, two and one, in a random manner, to either pole.

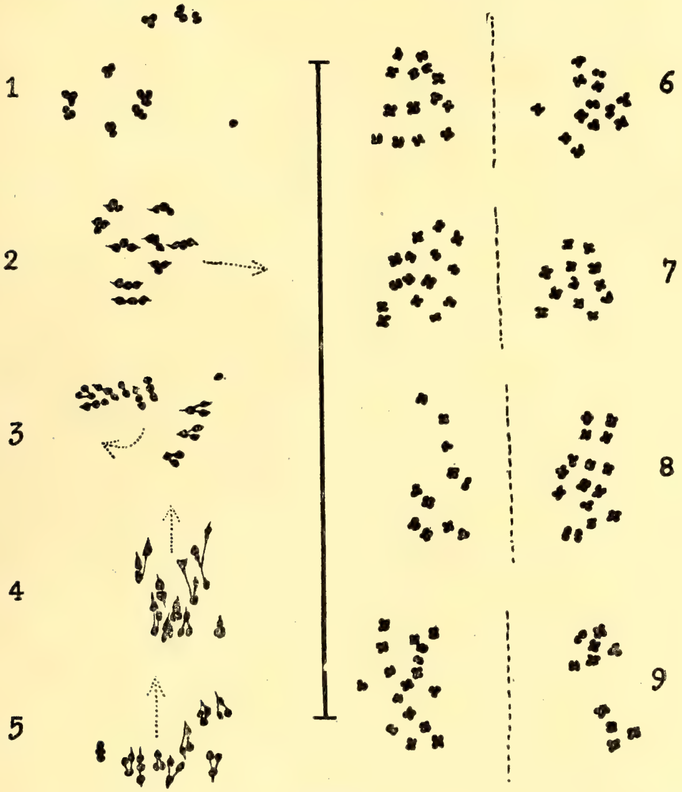

FIG. 2

The assortment of the chromosomes in the pollen-mother-cells of a triploid Canna. (1) Prophase showing triads. One chromosome separated from its corresponding dyad. (2), (3), (4), and (5) metaphase and early anaphase groups. In (3), one chromosome has again separated from its two partners. (6), (7), (8), and (9) metaphase plates for the second divisions, with cell-wall between, showing, respectively, 14 + 13, 15 + 12, 11 + 16, and 17 + 10 chromosomes.

In 1920 I grew 46 differently named clones and species of Canna, obtained, without particular selection, from three leading dealers of New York, Philadelphia, and Florida, and from the U. S. Department of Agriculture. The chromosomes of 31 of these were counted in the first or second divisions of the pollen-mother cells; about 250 groups being drawn with the camera. Of these 31 clones, 22 showed nine dyads or bivalents, which split into 9 + 9 at the first division (fig. 1); 3 clones had a total of 18 single chromosomes (or 9 bivalents), which divided at the first division into 8 + 10, or other unequal numbers, not commonly into 9 + 9; 5 clones were probably completely or nearly triploid, and irregular in their first division, resembling in this the triploid mulberries of Osawa,[5] and like them showing a smaller number of chromosomes after the first division than the triple number (in these cases, 24 to 26 instead of 27); while one clone was regularly triploid, showing nine triads at the prophase and first metaphase, and a total of 27 chromosomes after the first division.

This regularly triploid clone was obtained from Thorburn, New York, in 1920, under the name "Gladiator." It differed conspicuously from other Cannas with more than 18 single chromosomes, in its smaller flowers and lesser size, resembling in these the ordinary diploid Cannas. Thirty-two pollen-mother-cells were drawn with the camera. In 18 of these pollen-mother-cells, the total number of chromosomes could be accurately counted, and was 27. In the 3 other cases where the total number could be counted, some chromosomes seemed to be missing, for the totals were 25, 25, and 24. (In 3 pollen-mother-cells showing the anaphase of the first division, or the metaphase of the second, only one group could be accurately counted, because of the slanting position of the other. In the 8 remaining prophase or metaphase figures, not all the trivalent chromosomes could be distinguised from the bivalents or univalents into which they had divided, or from which they were composed.)

Nine of the cells showed clearly how many chromosomes went to one pole and how many to the other after the first division.

Chromosome partition	18 + 9	17 + 10	16 + 11	15 + 12	14 + 13
Nos. of cells found	0	1	2	2	4
Calculated on random distribution	0.04	0.3	1.3	3.0	4.4

(In the triploid Datura the chromosomes have been accurately counted in 64 pollen-mother-cells after the first division, and the distribution corresponds with a random one.)

Three of the early anaphase plates showed how the triads divided. Taking for "up" the upper side of the camera drawing, which was a random position, we have for the position of the two's with regard to the equator (fig. 2, Nos. 4 and 5):

	Up	Down	Doubtful
Third pollen-mother-cell	4	4	1
Second pollen-mother-cell	4	3	2
First pollen-mother-cell	2	4	3
Totals	10	11	

In several prophases, one chromosome was disconnected from its corresponding dyad (fig. 2, Nos. 1 and 3). If this occurred more frequently it would lead to the condition found in the large-flowered Cannas, where triads, dyads, and monads are mingled at the metaphase; or to the situation of Osawa's mulberry clones, where triads are apparently less common; or finally to the triploid Oenothera lamarckiana, which, according to Geerts,[3] shows only dyads and monads. In the triploid Canna, one complete count of both second divisions was made in a pollen-mother-cell. On one side of the cell-wall the numbers were 15 + 1 + 12, and on the other side, 13 + 13. The single chromosome seemed attached to the cell-wall.

About half of the pollen-grains of this triploid Canna were nearly or quite empty; while the others were full or nearly full of cytoplasm, with one or more nuclei.

Summary.—(1) Most of the Cannas examined were diploid, showing nine dyads before the first division in the pollen-mother-cells, and these in most plants separated into 9 + 9.

(2) One of the triploid Cannas showed commonly nine triads, each of which separated into two and one on the spindle, in a random manner with regard to the two poles. (The same arrangement, though less easy to demonstrate, was also found in a triploid Datura.)

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[2] Blakeslee, A. F., J. Belling, and M. E. Farnham, 1920, "Chromosomal Duplication and Mendelian Phenomena in Datura Mutants," *Science,* **52,** 388–390.

[3] Geerts, J. M., 1911, "Cytologische Untersuchungen einiger Bastarde von Oenothera gigas," *Ber. Deutsch. Bot. Ges.,* **29,** 160–166.

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[5] Osawa, I., 1920, "Cytological and Experimental Studies in Morus, with Special Reference to Triploid Mutants," *Bul. Imp. Sericult. Exp. Sta. Japan,* **1,** 317–369.

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[8] Tischler, G., 1915, "Chromosomenzahl,-Form, und-Individualität im Pflanzenreiche," *Prog. Rei. Bot.,* **5,** 164–284.

[9] Wilson, E. B., 1910, "Studies on Chromosomes VI. A New Type of Chromosome combination in Metapodius," *J. Exp. Zool.,* **9,** 53–78.

THE DESTRUCTION OF PHOSPHORESCENT ZINC SULFIDES BY ULTRA VIOLET LIGHT

By Leonard B. Loeb and Lloyd Schmiedeskamp

Ryerson Physical Laboratory, University of Chicago

Communicated by R. A. Millikan, June 12, 1921

It has long been known that the phosphorescence exhibited by certain alkaline earth sulfides is gradually destroyed when they are subjected to the action of alpha particles from radio-active matter.[1] It has also been found that a permanent destruction of the phosphorescent properties of such sulfides is caused by the bombardment of the sulfides with canal[3] rays or with cathode[2] and beta[1] rays. In this process of destruction, the sulfides are caused to phosphoresce brilliantly by the destroying agents, and the destruction is accompanied by a change in color of the sulfide, generally a darkening.

In 1919 J. Perrin[4] and more recently R. W. Wood[5] have shown that in the process of fluorescence, a process assumed to be somewhat similar to phosphorescence, the fluorescing molecules undergo chemical changes which result in the loss of the fluorescent power. In other words light of a given wave-length falling upon a molecule of fluorescent substance causes it to emit light of a different wave-length, and in this process the molecule is chemically changed so that it can no longer fluoresce.

The detection of the destruction of the fluorescent properties of certain substances is complicated by the fact that the fraction of molecules that are destroyed per second is relatively so minute that the time of exposure required to produce a measurable effect must be very long. As Perrin showed, this time may be much reduced by using very thin films of the solutions of fluorescing molecules. Wood has shown that it may also be much reduced by using intense sources of light. In view of the destruction of the phosphorescence of zinc sulfides by other agents which cause them to phosphoresce, and in view of the powerful technique developed by Perrin and Wood for the destruction of the fluorescent property by light, it occurred to one of the writers to attempt to detect a destruction of the phosphorescent property of the zinc sulfides through the action of light. The only reference to observations on such a destruction yielded by a survey of the literature consisted of the following quotation taken from a long paper by Baerwald[3] on the effect of canal rays on phosphorescence. Baerwald states "It is remarkable that intense ultra violet light from a mercury arc also destroys the phosphorescence of zinc sulfide. As it becomes dark in the process we may suspect a chemical change of the sulfide into an allotropic form, ..."

For the preliminary experiments which are being reported here, three different samples of phosphorescent zinc sulfides were used. One was a sample obtained from the Radium Standard Chemical Company in 1918, the others were two samples obtained in France from MM. Guntz and Ziegler of the powder works at Bouchet in 1918. The specimens to be studied were prepared by sifting the fine crystals upon small circular plane glass plates, 7 mm. in diameter which had been moistened with a very dilute solution of collodion in amyl acetate (about 3 drops of an ether collodion solution of the consistency of "New Skin" in 100 cc. of amyl acetate). Microscopic examination of the specimens showed that they were covered with a uniform layer of the sulfide occupying about 99% of the surface. The glass plate was fastened to a plate of thin transparent quartz, with the layer of sulfide pressed firmly against the quartz. The back side of the glass plate was then painted with optical black to permit the photometry of the sample. At the same time that the specimen was mounted on the quartz plate a control specimen as nearly the same as possible was similarly mounted on a glass microscope slide.

The quartz-mounted specimen was then placed in the image of a quartz Hareus mercury arc thrown upon the wall by a quartz lens of 3 cm. aperture and 15 cm. focal length. The arc operated continuously at about 2.5 amperes in a light-tight box. A blackened thermometer bulb placed in the image of the arc showed a rise in temperature of but 0.5° C. above that of the room, so that the heating of the specimen produced by the arc was not of importance. The glass-mounted control sample was kept in a light-tight box and was removed from this only at the time of photometric measurement.

The phosphorescent intensities of the quartz-mounted specimen and the control were compared by photometric measurements on the phosphorescence exhibited by them when studied in a simple phosphoroscope. The specimens were mounted on the window of a small photometer with the blackened side towards the window. This window was diffusely illuminated from behind by a small automobile headlight lamp. The intensity of illumination of the window was varied, and hence the photometry was accomplished, by changing the distance of the lamp from the window until the phosphorescent specimen faded against the window which served as its background. A color match between the lamp and the specimen was obtained by the use of absorbing screens of colored glass. In the photometry the specimens were excited to phosphorescence by the light from a 75-watt lamp focussed on them. The exciting light was put on and cut off in the usual way between the intervals of observation by means of slits in a metal disc rotated at a high speed, the procedure being in all respects the standard procedure used in phosphorescent measurements. After the quartz sample had been exposed to ultra violet light for a suitable interval of time, its intensity was compared with the control as above. The results obtained on three samples are tabulated below.

FRENCH SAMPLE NO. 2

Intensity	Time in hours	Date	Comments
1.00	0	3–21–21	
0.74	94	3–27–21	
0.76	159	3–30–21	
0.67	221	4– 2–21	
0.66	315	4– 6–21	
0.57	428	4–11–21	
0.69	527	4–14–21	Exposed to Cl_2 contamination in air of room
0.51	665	4–20–21	
0.95	689	4–22–21	Exposed to Cl_2 contamination in air of room
0.71	834	4–28–21	
0.84	858	4–29–21	After 24 hrs. in Cl_2 gas in flask

FRENCH SAMPLE NO. 3

Intensity	Time in hours	Date	Comments
1.00	0	3–25–21	
1.00	26	3–26–21	
.43	64	3–28–21	
.41	101	3–30–21	
.34	163	4– 2–21	
.23	257	4– 6–21	
.24	370	4–11–21	
24	466	4–14–21	Slight Cl_2 contamination in air of room
.27	604	4–20–21	
.49	628	4–22–21	Exposed to strong Cl_2 in air of room
.30	771	4–28–21	
.26	033	5– 9–21	

AMERICAN SAMPLE

Intensity	Time in hours	Date	Comments
1.00	0	3–30–21	
.84	95	4– 2–21	
.59	304	4–11–21	
.60	399	4–14–21	Slight Cl_2 contamination in air of room
.82	589	4–22–21	Exposed to strong Cl_2 contamination in air of room
.80	728	4–28–21	
.57	818	5–2–21	
.53	986	5–9–21	

In the first column are given the intensities of the specimens compared to the controls corresponding to the number of hours exposure to the ultra violet light recorded in the second column. In the third column are given the dates on which the measurements were made; while the fourth column is reserved for comments. In every case it may be seen that there is a marked reduction in the phosphorescent intensity with the increase in time of ultra violet illumination. In fact the French specimen No. 3 is seen to have dropped to a fourth of its initial intensity after 300 hours in the image of the arc. Though the changes are not very regular, the reduction in most cases is continuous with time, until the measurements made on the 14th of April are reached. The irregularities up to that date were doubtless due to the difficulties involved in the photometry of weak sources. On the 13th and 14th of April the air in the room in which the specimens were being exposed was mildly contaminated with chlorine gas used in the course of other experiments which were being performed in the same room. Again, from the 18th to the 22nd of April the air was contaminated by this gas, the contamination being specially strong on the 21st and 22nd. It is seen that the presence of the chlorine gas apparently arrested further decrease in intensity of two of the samples after April 14th; while it actually *increased* the intensity of the French Sample No. 2 which was subjected to a particulary heavy dose of the gas. On the 22nd, after the exposure to the strong chlorine, it is seen that all the samples showed marked increases in phosphorescent intensity. This was followed by a decrease in intensity on further illumination in the absence of chlorine after that date.

The reduction in intensity of the phosphorescence was in all cases accompanied by a darkening of the sulfide. Microscopic examination showed that the characteristic yellow color of the sulfides had disappeared and was replaced by a slight but uniform blackening throughout the crystal mass. This blackening was much like that of some glasses that have been subjected to cathode ray bombardment. To eliminate the possibility of this blackening having been caused by the collodion, a sample of the collodion film was exposed to the light for many hours. It showed no signs of blackening. A specimen of sulfide made up without the use of the collodion furthermore showed the same decrease of intensity and the same blackening on exposure to the ultra violet light as was shown by the samples using collodion. Following the accidental exposure to the chlorine gas it was observed that the specimens which had been blackened by exposure had regained their original color in a ring about 2 mm. wide at the edges of the specimens, i.e., where the chlorine had diffused inwards. It was these portions with restored color which showed the marked increase in intensity observed. The blackened centers of the specimens remained dull. Finally, to ascertain that it was the chlorine that caused the restoration of the color the French Sample No. 2 was placed in a flask with chlorine gas over night. On examination it was found to have regained its yellow color throughout its mass. On measurement, even though its edges were moist, due to the deliquescent nature of some of the reaction products, the intensity of the specimen was found to have increased markedly, as the table shows.

There is then little doubt in concluding from these experiments that the action of intense ultra violet light causes a destruction of the phosphorescent power of the zinc sulfides. This destruction is accompanied by chemical changes in the sulfides, resulting in a change of color, (possibly in a reduction or partial reduction of the polysulfides of the impurity (here probably Cu) required to produce the phosphorescent centers). It is further obvious that the exposure of the crystals to a powerful oxidizing gas such as chlorine at least in part restores both the original color and the ability to phosphoresce. It may accordingly be quite possible that the action of ultra violet light in this case is similar to that of ordinary light in the photographic processes, only much slower.

The results so far obtained hardly permit one to say much about the form of the curves of decay of phosphorescent intensity with the time of ultra violet illumination. They indicate, however, that the curves may either be the sum of a group of curves having the form $I = \frac{I_0}{At}(1 - e^{-At})$ given by Rutherford[6] for destruction of phosphorescence in the case of alpha particles where the intensity of the alpha particle radiation falls off exponentially with the distance below the surface; or they may be the sum of a group of exponential curves having different constants. Such curves would be expected in the case of measurements made on the decay of *total* phosphorescence with ultra violet illumination, for it is known that the phosphorescent light is composed of groups of different phosphorescent wave-lengths, each group probably having its own susceptibility to destruction, and hence each having its own time rate of destruction by ultra violet light.

The destruction of the phosphorescent power of these sulfides by ultra violet light noted above might point to a close analogy between the phenomena of phosphorescence and fluorescence in the light of the recent work of Perrin. Such an analogy would be in accord with the conclusion arrived at by Rutherford[6] in the case of the destruction of the phosphorescence by alpha particles, viz., that the light emitted under alpha particles bombardment *accompanies* the dissociation and *destruction* of the active centers. If this interpretation is correct, and if phosphorescence and fluorescence are closely related, the theory of Schmidt and Wiedemann,[7] Lenard,[7] and Merritt,[7] explaining the mechanism of phosphorescence will have to be modified. This theory ascribes phosphorescence to the return of ionized portions of the active centers (presumably electrons), to the centers from which they were removed by the exciting agent. Such a change does not involve a *permanent* destruction of the active center. According to Perrin the active center is permanently chemically changed in the process by which the substance emits its fluorescent radiation. It therefore follows that unless the production of phosphorescent light and the destruction of the active centers by the ultra violet light are two independent phenomena the existing theory must be altered. Further experiments are needed to decide this point. The writer's thanks are due to Prof. R. A. Millikan for his kind interest and criticism.

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*ACOUSTIC PRESSURE DISTRIBUTIONS, CHIEFLY IN RESERVOIRS AND IN PIPES**

BY CARL BARUS

BROWN UNIVERSITY

Communicated May 12, 1921

Introductory.—In *Science* (May, 1921) I described an available method of pressure measurement in connection with acoustic phenomena, the method consisting essentially of a shallow mercury U-tube, the head of which is read off by displacement interferometry. Thus the fringe displacement s (occurring throughout the following paragraphs) will vary directly as the head in question and therefore with the pressure in the region under investigation. When the shank of the U-tube communicating with the sounding region is quite closed or quite open, significant results are either absent or without interest; but when the sound leak is reduced to the dimensions of a pinhole in foil, or the like small aperture, relatively large pressure increments or decrements are usually found to reside in the vibrating medium, varying with frequency and particularly marked when the region is in resonance with the sound generator. As an instrument for this purpose, a telephone, actuated by a small induction coil and a motor current interruptor is preferable; since non-symmetric inductions conduce to greater sharpness and strength in the phenomena to be observed. The current so obtained may be reversed, changing all the pressure increments into corresponding dilatations. The effect of the symmetrical induction of the magneto does not admit of commutation;

*From a forthcoming Report to the Carnegie Inst., Washington, D. C.

but it is nevertheless definite. When a large resistance, r, put in the telephone circuit produces a fringe deflection, s, the equation sr = constant is near enough for practical purposes and thus the passage from s to p (when called for) is very convenient.

It is the object of the present paper to continue these experiments by treating pressure distributions in vibrating regions of different size and in slender pipes. The case of wide organ pipes I have also investigated, but the results must be given later.

In figure 1, the two shanks of the mercury U-tube at least 6 cm. in diameter (R, air volume 48 cm.^{3}, R', air volume 135 cm.^{3}), are closed on top by glass plates, to receive the rays of the vertical interferometer; i.e., the component rays enter R and R', normally to the diagram.

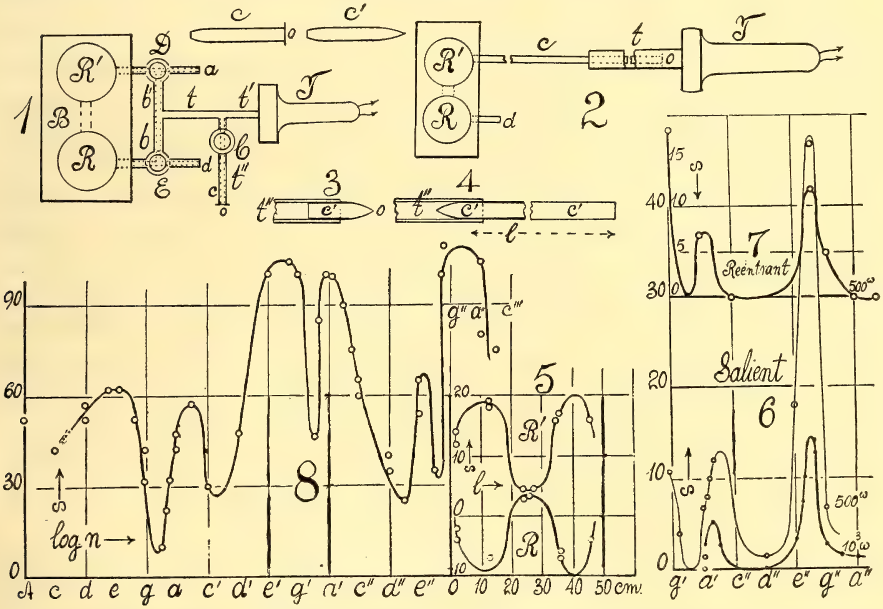
U-tube Used Differentially.—The two different reservoirs R and R' of the U-tube, figure 1, were placed in communication by the branch pipe b, b', containing the three way cocks, D, E, shutting off the air vents a, d, and leading to the telephone T through the quill tubes t and t'. The pinhole is in the second branch t'' at O and a cock, C (here open), is interposed. I had anticipated a differential fringe deflection, owing to the different resonance volumes at R and R'. The result, however, was absolutely negative. Each reservoir (R, R') thus acts like an open air communication with reference to the other, so that the effect of the pinhole valve O vanishes. On using either side separately ($t\ b\ R$ closed, $R'\ D\ a$ open to the atmosphere, or $t\ b'\ R'$ closed, $R\ E\ d$ open) the normal behavior at once appeared.

The only way of securing a differential effect detected, was by elongating either branch (b, for instance, by inserting a long piece, 40 to 80 cm., of rubber tubing). No doubt, this is merely equivalent to stopping, partially, access to either chamber, R or R'.

A variety of correlative experiments were made with the simple (non-differential) apparatus ($t\ b\ R$ closed $R'\ D\ a$ open). I have elsewhere referred to the absence of fringe displacement (nearly) when the copper foil carrying the pinhole is cemented to the mouth of a funnel tube; but the prolongation of the quill tube t'', if the diameter is not increased, is almost indefinitely permissible.

The experiments made suggest a method of obtaining an effect which is at least apparently differential. For this purpose the cock C (fig. 1) is to be closed, so that c is inactive, and the pinhole c (fig. 1) to be inserted either into the branch b' or b. With a normal fringe displacement of 40, the pinhole in b' gave a displacement of 38 fringes; in b correspondingly 27 fringes. Hence the large volume R' is more favorable to a larger displacement than the smaller volume R. It will be observed at once that here these reservoirs act merely like the outside air in the case of the original experiment. Thus there is no true differential effect resulting from the size of resonators, whether the region be closed, open or partially open.

Conical Vents Reversible. Periodicity.—The conical vents c' (fig. 1) may be inserted into the branch tube t'' in two ways, either in the salient position a with the convex surface around the pinhole O outward (fig 3); or in the reentrant position (fig. 4) with the convex surface at O inward. The results obtained are usually reversals of each other, so that a pressure excess is liable to be on the concave side of the conical vent. These results were always consistent in character; but it was soon found that the strength of the telephone current and the length l of the tube $c'c'$ in the reentrant position (fig. 4) greatly modified them. When there is no resistance in circuit, i.e., when the telephone sounds harshly, reversal ceases, so that either the case 3 or 4 produces a pressure within; but even here, on closing the circuit, the fringes in case 4 are seen to move first toward a dilatation and then turn in the direction of pressure.



The position (fig. 3) being the normal case investigated above, the case 4 was studied with 1,000 ohms in circuit, and for different lengths l of the quill tube c', beyond O, from l = 2 to over 40 cm. These results are given (fig. 5) the abscissas showing the length l of tube taken and the ordinates the fringe displacements s, both for the region R' alone (positive displacements here denoting dilatation) and for the region R (positive displacements denoting pressure). The results are periodic in marked degree, so that the quill tube (fig. 4) is a musical instrument with a pinhole embouchure; and in fact while the case 3 is nearly silent, 4 audibly reproduces the sound of the telephone. The curve for R' shows two resonance maxima and one minimum, but in all cases the dilatation (positive-displacement s) within R' is sustained, merely changing in degree. The curve for R, however (dilatations for negative s) indicates the occurrence of both dilatations and pressures within this (smaller) region. Both curves are quite consistent (although R' is nearly four times more capacious than R) and one may infer the length of pipe l = 30 cm. (fig. 5) or the wave length $2l$ = 60 cm. to be an harmonic of the telephone interruptor. This was in fact close to the four foot c.

In a majority of cases the action of the conical vent thus recalls the behavior of the cup anemometer, as the pressure excess is on the concave side; but the lower curve of figure 5 (for R smaller) is out of keeping with this, as between lengths of 20 and 35 cm. of pipe, the pressure excess is within, or on the salient side.

Resonators of Very Large Capacity.—The volume of the region R' (fig. 1) was now further increased by adding an additional cylindrical tube 6.2 cm. in diameter and 10.7 cm. high, closed on top with a glass plate. All parts were (as before) carefully cemented together. The volume added was thus 370 cm.^{3}, as compared with the original 48 cm.^{3}, the ratio of volume increment being 6.7, and the ratio of total to original volume 7.7 The conical vent c', figure 1, here acted much better than the copper foil pinholes c.

Using the motor break and conical valve, the fringe displacements were observed for frequencies between the notes g' and c''' and both with 1,000 ohms and 500 ohms in the telephone circuit. An example of the results is given in the two curves in figure 6, the region R being in communication with the atmosphere and R' closed except as to the salient conical vent specified. The curves are remarkable because of the sharpness of the maxima, which are apparently overtones in the key of B or B flat. It is obvious that the fundamentals of the large closed reservoir R' will lie very low as compared with the frequencies of the diagram and very large fringe displacements may be looked for and were found there. The height of the maxima gradually decreased from g to c''' there.

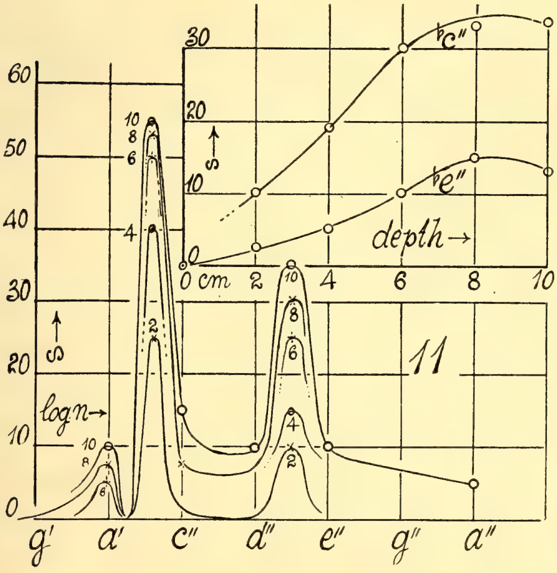
It was now thought desirable to test the conical vent in the reentrant position and data of this kind are given by the curve in figure 7. All the maxima are here dilatations, laid off positively for convenience in comparison with figure 6. The valve action (500 ohms in circuit) is much weaker in figure 7 than in figure 6; but in every other respect figure 6 is symmetrically reproduced. This is a very disconcerting result; for it is not the impulsive displacement of the telephone plate alone which produces the pressure increments within, even if a reversal of the telephonic current (change of poles) changes pressure increments into pressure decrements. The reversal conical pinhole vent in association with a given adjustment of telephone current (poles not changed) will do the same.

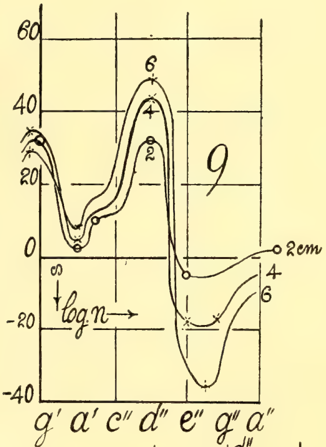
Resonator of Very Small Capacity.—Finally the resonator R' was all but closed (resonator IV), or at least reduced to a shell-like space by a cylindrical inset, closed with a glass plate but a few millimeters above the mercury of the U-tube.

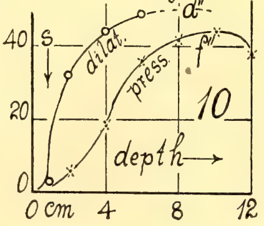
A survey of the fringe displacement corresponding to different harmonics is given in figure 8, 1,000 ohms completing the telephone circuit. Their distribution is less regular than heretofore, and probably the smaller serrations have escaped detection. What particularly astonishes is the occurrence of resonance at the low notes, seeing that the resonator volume is here relatively negligible. The maximum displacement of 100 fringes corresponds to a pressure increment of about .03 cm. of mercury, which though twice as large as obtained with the large resonator III, is nevertheless of the same order of values.

In one respect the present curve (fig. 8) for the shell resonator IV, differs radically from the complete curve for the capacious resonator III. In the latter, conformally with the large volume, the lower notes (8-foot octave) are very much more effective than the notes of the one foot octave. The curve as a whole falls from left to right. In figure 8 the reverse tendency is observed.

The question as to what resonates in case of the small resonator was approached by lengthening the tubing between the telephone and U-tube from its usual length of 48 cm., to 85 cm. and 120 cm. successively. The







shell-like volume of the resonator IV cannot much have exceeded 10 or 15 cm.^{3} The tubes added 10, 17, and 24 cm.^{3}, respectively. To this the shallow volume in the telephone (about 2 cm.^{3}) is to be added; nevertheless the greater part of the total volume seems to reside in the tubes.

The results obtained in the frequency survey for maxima and minima showed that the overtones had been differently accentuated. Thus the minimum near d'' passes from positive to negative values as the tube length increases, while the reverse is observed at the maximum near g''. The case for tube additions is then analogous to the occurrence of the vowel sounds, where different overtones are accentuated by variation of the mouth cavity.

The Pinhole Sonde, or Probe.—The possibilities of sounding for pressure by aid of a pinhole valve, O, at the end of a slender tube, c (as in figure 2), communicating with the closed reservoir R' of the U-tube, the other, R, being open to the atmosphere are next to be treated. The telephone T was at first provided with a tubular projection t, 12 cm. deep to the plate and about .75 cm. in inner diameter. This received the probe c (O being at the end of the aluminum tube C 2 mm. in bore, 3 mm. in outer diameter and about 20 cm. long) to different depths, with a view to exploring the pressure within.

The results obtained on inserting the probe to different depths (marked in the graphs, data for 8, 10, 12 cm. of depth showing no further change in the telephone pipe) are given in figure 9, pressures laid off positively downward. The curves are unexpectedly complicated, but consistent, in their character throughout. In place of the simple distribution of pressure in the usual closed organ pipe, there is a maximum pressure increment near f', another at d'' and a very pronounced pressure decrement (negative fringe displacement) near f''.

The increase of pressure from the mouth of t toward the telephone plate 12.5 cm. below, is well shown in figure 10, in which the fringe displacements s for the large dilatation d'' are recorded. The other curve gives the distribution of pressure at f''. These increments rise very rapidly from the mouth inward and at 8 cm. of depth already reach a maximum value. Beyond this, the curves like figure 10 are apt to show a decrease, which is not probably incidental.

The rubber tube was now about halved in length, the present depth to plate being about 5 cm. and the diameter .75 cm. The note obtained with the blower was a faint d' and a strong g''. In the survey in pitch with the pinhole at a depth of 4 cm. there were maxima at d' and particularly at the octave of the blown pipe d'', and minima at c' and a', all well marked. On comparison of the curve for the half pipe with figure 9, for the full pipe, there was actually a complete inversion of results, maxima taking the place of minima. The largest displacement occurred at about 5 mm. from the plate.

Pressures in Smooth Straight Pipes.—These surveys in pitch and depth were completed for a number of telephone blown brass pipes, 1 cm. in diameter and of different lengths; but the very interesting graphs obtained must be omitted here. Thus the pipe 13 cm. long showed enormous maxima at a' and correspondingly large negative minima at d'', at all depths (2, 4, 8, 12 cm.) below the mouth of the tube. Figure 11 reproduces the behavior of the same tube cut down to 10 cm. of length, at different depths 2, 4, 6, 8, 10 cm. below the mouth. Finally similar graphs for a variety of wide tubes and resonators have been worked out and progress made with the installation for symmetrical induction.

THE PREDICTION OF ANNUAL EGG PRODUCTION FROM THE RECORDS OF LIMITED PERIODS

By J. Arthur Harris, W. F. Kirkpatrick and A. F. Blakeslee

Station for Experimental Evolution, Carnegie Institution, and the Storrs Agricultural Experimental Station

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For the past several years the writers have been considering the possibility of predicting the annual egg production of the domestic fowl from the records of short periods of time. Such records may be determined by trap-nesting, or by the use of other criteria when the maternity of the eggs is not required for breeding purposes.[1]

The first definite step in the direction of the use of the egg record of a short period for the prediction of the production during a subsequent or a longer period was, as far as we are aware, taken in 1917 when it was shown[2] that in a heterogeneous series of birds such as are submitted by practical breeders in egg laying contests, the October egg production is correlated with that of every other month of the year. The investigation was carried much further in a second memoir[3] in which the correlations between the records of the individual months and the production of the whole year, between the records of the individual months and those of the remaining 11 months of the year, and between the production of 5 of the individual months and the production of all the other individual months, were published for two series of birds. In this paper the equations for the prediction of total annual production from the record of the individual months were given.

Our purpose here is to state briefly the results of a first test of the possibility of utilizing the linear regression equation (which is strictly valid only for the population from which it is deduced) for the prediction of the records of the birds of a flock the performance of which is unknown as far as the determination of the constants of the equations is concerned.

In a population the straight line relating the egg production of a period E_p with that of a period used as a basis of prediction, e, is

$$E_p = \left(\bar{E}_p - {}^{r}eE_p \frac{\sigma E_p}{\sigma_e}\, \bar{e}\right) + {}^{r}eE_p \frac{\sigma E_p}{\sigma_e}\, e$$

where the bars denote means, the sigmas represent standard deviations, and r indicates the correlation of the two variables in the standard population.

The value of E_p given by the equation is the theoretical mean production for the array of individuals of any class with respect to e. The assumption to be tested is that we may write E_p' and e' instead of E_p and e, where E_p' is the theoretical mean production for a period p of the array of birds of any grade of production e in the period used as a basis of prediction in a series of birds which are not involved in the data upon which the equations were based, but to which the equations are to be applied for practical purposes.

The essential practical requisites for such prediction equations are: (1) That the errors of prediction shall be distributed about the true numbers in such a manner that estimation will not in the long run be either too high or too low. (2) That the magnitude of the deviation of the predicted from the observed egg production shall be as small as possible.

Let E_p'' be the actual and E_p' the predicted egg production of an individual bird for any period, p, in a flock to which the equation is being applied. The error of prediction is then $E_p' - E_p''$. The average of these errors, with regard to sign

$$\frac{1}{N}\Sigma\left(E_p' - E_p''\right)$$

furnishes a measure of the success with which the first requirement, (1) above, is met. The average of these errors without regard to sign furnishes a measure of the average error above or below the true production of the individual birds of a flock. The square root of mean square deviation

$$\left[\frac{1}{N}\Sigma\left(E_p' - E_p''\right)^2\right]^{1/2}$$

furnishes a measure of this error which weights larger errors.

The errors may be expressed in actual numbers of eggs, or, in relative terms, as percentages of the mean production of the period and flock for which prediction is made. Both methods have been used in testing the equations.

In testing the efficiency of such equations for purposes of prediction we have proceeded in a purely objective manner. Working on the assumption that the crucial test of any theory is its capacity for predicting the unknown, we have calculated equations based upon the data of the International Egg Laying Contest at Storrs, Conn., during the six contest years, 1911–1917, inclusive. We have then used these equations to predict the annual production (and the production of groups of months) for the birds of the 1917–'18 contest, using as a basis of prediction the individual months of the laying year separately, pairs of successive months and groups of three months. Our conclusions concerning the value of the equations depend, therefore, not upon *a priori* considerations but upon the results of actual tests of accuracy of prediction for series which were unknown as far as the determination of the constants of the equations is concerned.

Consider first of all the results of the attempts to predict the annual egg production of 415 White Leghorn birds observed at Storrs from Nov. 1, 1917 to Oct. 31, 1918 from the records of a single month's production. The results of the three criteria of accuracy of prediction are summarized in table 1.

TABLE 1

ERRORS OF PREDICTION OF ANNUAL EGG PRODUCTION FROM THE RECORDS OF INDIVIDUAL MONTHS

MONTH USED AS BASE OF PREDICTION	AVERAGE DEVIATION WITH REGARD TO SIGN		AVERAGE DEVIATION WITHOUT REGARD TO SIGN		SQUARE ROOT OF MEAN SQUARE DEVIATION	
	Actual deviation	Percentage deviation	Actual deviation	Percentage deviation	Actual deviation	Percentage deviation
November	+ 2.39	1.52	29.59	18.78	38.65	24.52
December	— 0.49	0.31	29.26	18.57	37.61	23.86
January	+ 2.58	1.64	30.09	19.09	38.77	24.60
February	+ 0.06	0.04	27.28	17.31	34.70	22.02
March	— 1.63	1.03	27.95	17.73	34.28	21.75
April	— 6.23	3.95	28.72	18.22	35.31	22.40
May	+ 7.02	4.45	28.62	18.16	35.89	22.77
June	— 5.21	3.31	29.03	18.42	36.53	23.18
July	— 5.27	3.34	28.35	17.99	35.89	22.77
August	— 0.82	0.52	26.87	17.05	34.34	21.79
September	+ 4.78	3.03	24.78	15.72	32.94	20.90
October	+ 3.95	2.51	27.37	17.37	36.47	23.14

Considering first of all the absolute values we note that the average errors with regard to sign are generally low. Thus the prediction from November and from January production gives on the average 3 eggs too many for the year. For December, February, March and August the prediction is in error by less than 2 eggs. The values predicted from April, May, June, July, September and October records are from 4 to 7 eggs in error.

The average deviations without regard to sign are of course much larger since they constitute a measure of the error of prediction of the records of individual birds. They range from 24.8 to 30.1 eggs. The significance of errors of this magnitude will be more clearly brought out later.

The square root of mean square deviation also shows considerable regularity from month to month. These measures are naturally considerably larger than the average deviation without regard to sign. They range from 32.9 to 38.8 eggs.

It is clear that the annual egg production of birds similar in origin to the series upon which the prediction equations were based and maintained under similar conditions may be predicted with a relatively high degree of accuracy providing their record for any month is definitely known.

The order of the errors will be more readily understood by expressing them in relation to the average production of the flock, as shown by the percentage deviations.

We note that in predicting from December, February and August records the average error with regard to sign *is less than one per cent of the average annual yield of the flock.* In predicting from November, January and March the error lies between one and two per cent. When April, May, June, July, September and October records are used as a basis of prediction the average errors of prediction are from 2.50 to 4.50 per cent of the average annual yield.

The average deviations without regard to sign are less than 20 per cent of the annual production. The values for the individual months range from 15.7 for September to 19.1 for January.

The square root of mean square deviation is less than 25 per cent of the average annual production. The individual values range from 20.9 for September to 24.6 for January.

These two latter tests may at first seem to indicate very unsatisfactory prediction. Such is not, however, the case. These give the average errors *either above or below the true record made in the prediction of the results for an individual bird.* The thing which is required in practise is generally *the prediction for a group of birds of a particular grade* of egg record for the month used as a base of prediction. In a flock of 415 birds this has been shown to be possibe *with an error of less than 5 per cent of the annual production* when prediction is made from the record of *any* month of the year; and *with an error of less than 1 per cent when prediction is based upon the records of a number of the individual months.*

Lack of space precludes a discussion of the results of the prediction of the annual record of the bird from the combined record of two consecutive months. We may, however, illustrate the accuracy of prediction from the combined record of two consecutive months by means of the figures in diagram 1 which shows the accuracy of prediction from November plus December and from April plus May in comparison with the results of prediction from November and April. In these the estimated production is shown by a straight line.

The actual production for the year for which prediction is made is shown by solid dots for each group of birds as classified by monthly or bimonthly

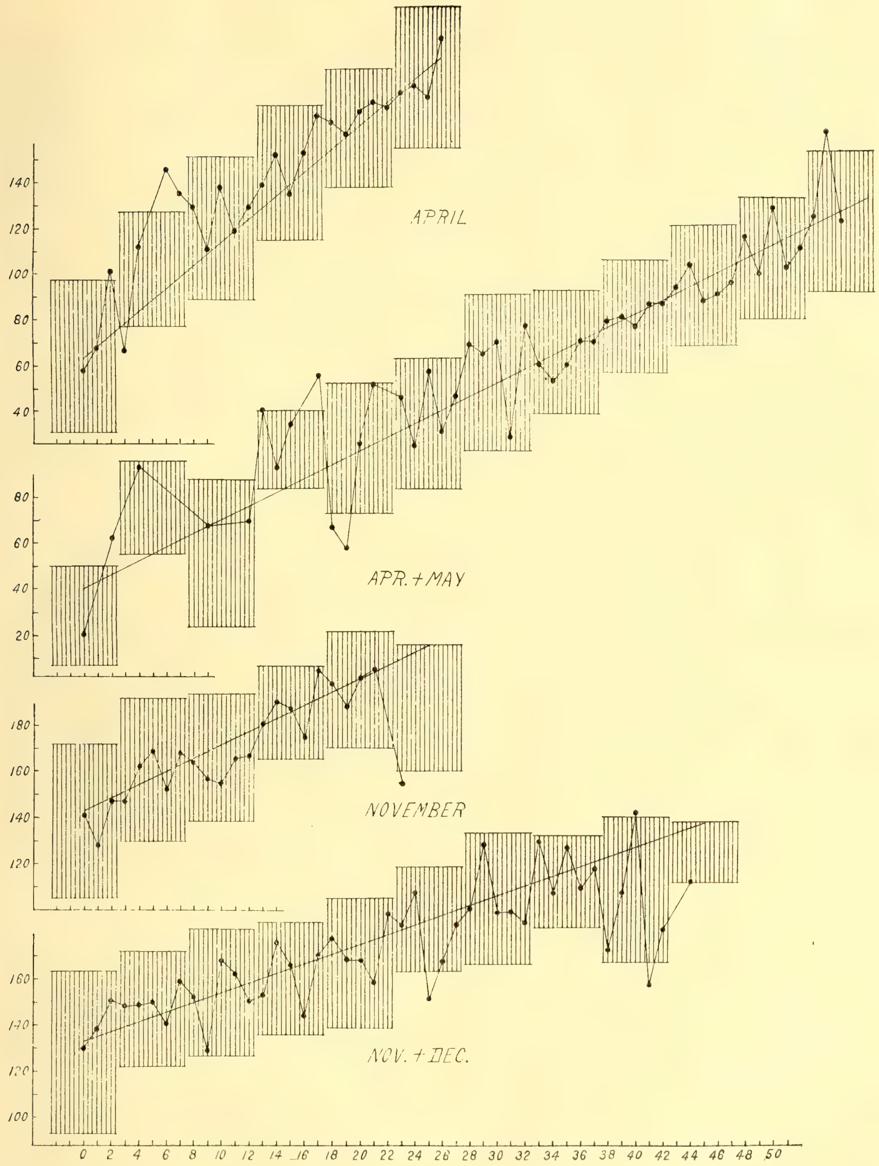

DIAGRAM 1

record. The shaded areas are determined as follows. The birds were first grouped into classes of five eggs range with respect to number of eggs laid during the period of time used as a basis of prediction. The birds of these classes of five eggs range were further subdivided into those in which actual egg production was greater than the predicted and those in which the actual number was less than the predicted number.[4] The average error of prediction was determined for each of these groups, and these averages represent the upper and lower limits of the shaded areas. The upper limit represents, therefore, the average deviation (for the period for which prediction is made) of all birds which make a higher record than that predicted for their class. The lower limit of the shaded area marks the average deviation for all birds which show an egg record lower than that predicted. These diagrams, which are quite typical of the whole series, certainly indicate excellent prediction.

The results for the combined records of three consecutive months are shown in table 2. These show that greater accuracy of prediction may be obtained when the records of three months are used as a basis of prediction. Such a result is to be expected on *a priori* grounds. A careful comparison of the constants in tables 1 and 2 will show, however, that the improvement resulting from the trebling of the number of months used as a basis of prediction is not great.

TABLE 2

ERRORS OF PREDICTION OF ANNUAL EGG PRODUCTION FROM THE RECORD OF THREE CONSECUTIVE MONTHS

THREE MONTHS USED AS BASE OF PREDICTION	AVERAGE DEVIATION WITH REGARD TO SIGN		AVERAGE DEVIATION WITHOUT REGARD TO SIGN		SQUARE ROOT OF MEAN SQUARE DEVIATION	
	Actual deviation	Percentage deviation	Actual deviation	Percentage deviation	Actual deviation	Percentage deviation
Nov.–Jan.	+ 2.09	1.33	25.93	16.45	33.84	21.47
Dec.–Feb.	+ 0.78	0.49	25.31	16.06	32.65	20.72
Jan.–Mar.	+ 0.49	0.31	25.29	16.05	31.58	20.04
Feb.–Apr.	— 4.07	2.58	24.16	15.33	29.77	18.89
Mar.–May	— 0.73	0.46	25.42	16.13	31.14	19.76
Apr.–June	— 2.31	1.47	24.33	15.44	30.59	19.41
May–July	— 2.12	1.35	24.20	15.36	29.40	18.65
June–Aug.	— 5.35	3.39	23.49	14.90	29.80	18.91
July–Sept.	— 0.20	0.13	21.36	13.55	28.10	17.83
Aug.–Oct.	+ 3.91	2.48	21.59	13.70	29.23	18.55

Prediction of the number of eggs which will be laid in the period subsequent to the month or group of months used as a basis of prediction may also be made. The errors for such a series of predictions, in which each individual month of the year (with the exception of the final month) has served as a basis for the prediction of the egg production of the remaining months of the year, are shown in table 3. The constants in this table show that when the period for which prediction is made is a long one a degree of accuracy fairly comparable with that for the whole year is attainable. The absolute values of the average deviation without regard to sign and of the square root of mean square deviation necessarily become smaller as the period for which prediction is made becomes shorter. The relative (percentage) error, however, increases. Thus the accuracy of prediction decreases rapidly as the period for which prediction is made becomes shorter.

TABLE 3

Errors of Prediction of the Record of a Period of Months from the Record of Individual Preceding Months

PERIOD FOR WHICH PREDICTION IS MADE	MONTH USED AS BASE OF PREDICTION	AVERAGE DEVIATION WITH REGARD TO SIGN		AVERAGE DEVIATION WITHOUT REGARD TO SIGN		SQ. ROOT OF MEAN SQ. DEVIATION	
		Actual deviation	Percentage deviation	Actual deviation	Percentage deviation	Actual deviation	Percentage deviation
Dec.–Oct.	November	+ 2.39	1.57	29.59	19.49	38.65	25.46
Jan.–Oct.	December	+ 0.24	0.16	28.43	19.53	36.63	25.16
Feb.–Oct.	January	+ 2.37	1.71	26.48	19.06	34.61	24.91
Mar.–Oct.	February	+ 0.09	0.77	24.07	18.66	30.86	23.92
Apr.–Oct.	March	— 0.24	0.21	22.81	20.36	28.40	25.34
May–Oct.	April	— 4.00	4.23	21.36	22.59	26.50	28.02
June–Oct.	May	+ 3.62	4.98	19.89	27.35	24.81	34.11
July–Oct.	June	— 2.86	5.34	17.62	32.91	21.44	40.04
Aug.–Oct.	July	— 3.71	10.43	13.91	39.09	17.14	48.17
Sept.–Oct.	August	— 2.56	13.57	9.76	51.75	12.12	64.26
October	September	— 0.45	7.67	4.57	77.85	5.71	97.27

The results of this investigation, taken as a whole, show that in the case of a flock of White Leghorn fowl which is essentially identical in genetic composition and maintained under essentially uniform conditions from year to year it is quite possible to estimate annual egg production from the record of either a single month or of two or three consecutive months with a high degree of accuracy. The same is presumably true of other breeds as well. This point is now under investigation.

It is not possible to use the equations given in this paper for flocks differing greatly in genetic composition or in conditions of maintenance from that upon which these equations were based. The problem of the determination of corrective terms to be used when the equations are applied to flocks other than that upon which they are based is now under investigation.

A detailed account of these investigations is now in press in *Genetics*.

---

[1] Alder and Egbert, *Bull. Utah Agr. Exp. Sta.*, No. 162, 1918.

[2] Harris, Blakeslee and Warner, These Proceedings **3**, 1917 (337–341); Harris, Blakeslee, Warner and Kirkpatrick, *Genetics*, **2**, 1917 (36–77).

[3] Harris, Blakeslee and Kirkpatrick, These Proceedings **3**, 1917 (565–569); *Genetics*, **3**, 1918 (27–72).

[4] A range of five eggs was used in order to obtain a number of birds sufficiently large to reduce somewhat the irregularities due to the errors of random sampling. The errors of prediction were in each case determined for classes of unit range. Grouping is used for graphic representation merely. The average deviations represented by the limit of the shaded zone are to be thought of as measured from a line perpendicular to the ordinates and intersecting the prediction line on the mid-ordinate of the 5-egg class.

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SOME PHYSICAL CHARACTERISTICS OF THE EAR[1]

BY JOHN P. MINTON
RYERSON PHYSICAL LABORATORY, UNIVERSITY OF CHICAGO

Communicated by R. A. Millikan, June 3, 1921

By means of the audion oscillator, a special telephone receiver tuned to a high natural period and placed against the ear and a suitable circuit numerous tests at various frequencies have been made on the minimum audibility current for both normal and abnormal ears. From these data important conclusions can be drawn regarding the physical and physiological structure of the ear and also concerning the functioning of the ear and the auditory nerves with their endings as organs of hearing. The conclusions drawn, then, are based on precise physical measurements which can be checked to within about five per cent from time to time.

Sensitivity of Normal Ears.—Ears are considered normal when no physiological defects are observable by the otologist. If we define sensitivity as the reciprocal of the minimum vibrational energy in ergs of the receiver diaphragm that the normal ear can detect, then curves can be plotted showing the sensitivity as a function of the frequency. This has been done for a number of normal ears. The curve shown in figure 1 is a typical one. It is to be observed that up to about 6000 cycles three distinct maxima of sensitivity are present; one at 900 cycles, one 1800 and another one at 3900. It is also important to observe that the sensitivity is much greater throughout the region from 200 to 4500 cycles than outside this range. Within this region are included all the frequencies which are of most importance for both speech and music. The natural period of the receiver was 5215 cycles when this curve was taken and the absence of a peak in this position shows that the receiver characteristics have been corrected for satisfactorily. All the maxima, then, occur below the resonant frequency of the receiver.

Curves on other normal ears show maxima which agree for the most part in location but not in magnitude with the curve shown in figure 1. Because of the marked similarity between the response curves for coupled mechanical vibratory systems and the curves for normal ears, it seems

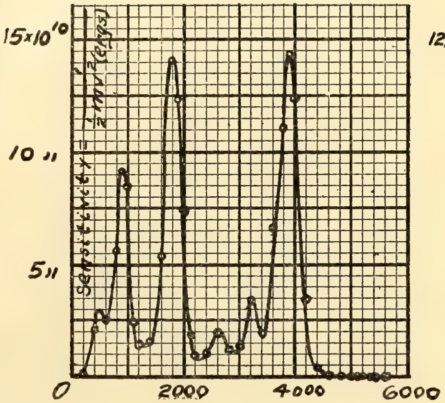


FIG. 1
Frequency sensitivity curve for normal reference ear.

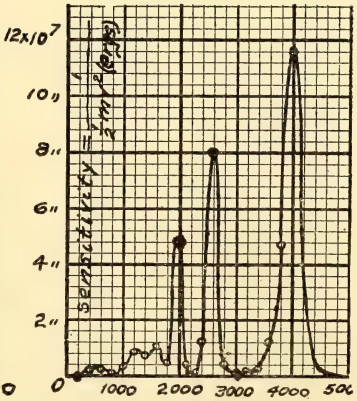


FIG. 4
Frequency sensitivity curve for ear referred to in figure 2.

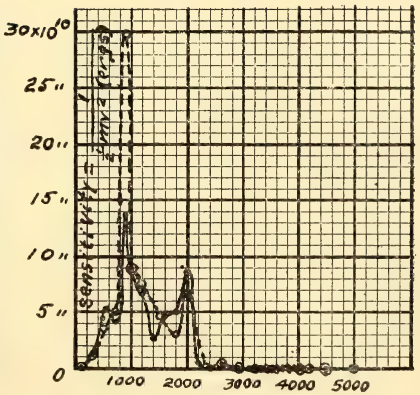


FIG. 5
Frequency sensitivity curve for ear referred to in figure 3.

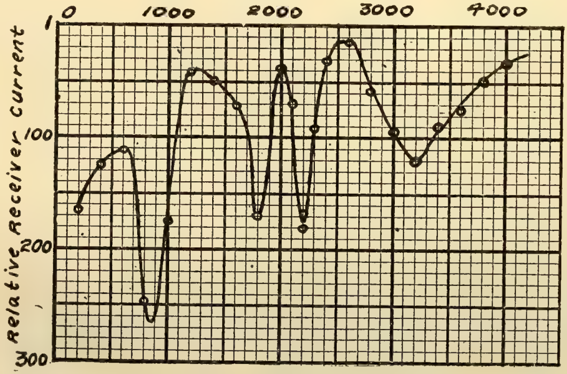


FIG. 2
Frequency. Relative to normal reference ear.
Middle ear trouble.

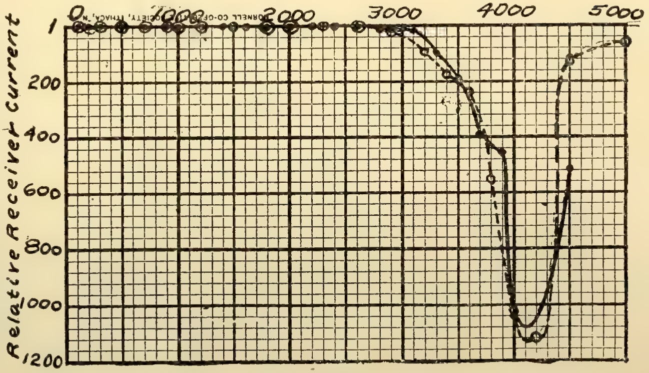


FIG. 3
Frequency. Relative to normal reference ear.
Internal ear trouble.

as though we are justified in thinking of the ear as whole as a coupled vibratory system and referring to the maxima of sensitivity as its resonant frequencies. The resonant frequencies are not to be thought of as harmonics of the fundamental.

Sensitivity of Abnormal Ears.—In the study of abnormal ears, where disease has affected certain portions of them and has produced partial deafness, curves of two types are plotted. The first type shows at the various frequencies how much current must pass through the receiver for the patient to just hear the tones compared with that required for a normal ear to hear. The second type shows the sensitivity of the ear at the various frequencies. Both methods of plotting the data are of value for studying the physical characteristics of the ear.

In figures 2 and 3 are shown curves relative to the normal reference ear for two totally different types of deafness. The former curve is for a case of oto-sclerosis (so-called "Fixation of the Stapes"), a type of middle ear deafness and the latter curve is for a case of internal ear trouble (so-called "Nerve Deafness"). The curves are alike in no respects, whatsoever. In the case of middle ear trouble there are marked depressions relative to the normal reference ear (figure 1) in the regions of 600 to 1200, 1500 to 2000, 2000 to 2500 and 3000 to 3500 cycles. There have been taken for different patients with middle ear deafness several curves and all of them agree in the location of these depressions. Some of these depressions are relative and some of them are absolute as reference to figure 4 will show. This figure (4) gives the sensitivity at the various frequencies of the ear referred to in figure 2. The two curves shown in figures 2 and 4 together give one accurate information on the physical characteristics of the defect. The stapes is not fixed but simply hindered and modified in its motion so as to produce less sensitivity than is possessed by a normal ear and also so as to give a sensitivity curve with three distinct resonant frequencies all of which are at higher frequencies than the corresponding ones for the normal reference ear and, as is to be expected, the peaks at the lower frequencies are displaced toward higher frequencies more than those in the upper part of the scale.

The curves for the case of the internal ear trouble shown in figures 3 and 5 are of much value in suggesting how the internal ear system functions physically. The patient's hearing is essentially the same as a normal ear up to about 2600 cycles as shown by figures 3 and 5; shown better in the latter, however. Then, the depression (shown better in figure 3) from 2600 to 4500 cycles is very great. When one recalls that the vibrational energy of the receiver diaphragm is proportional to the square of the receiver current, he will then grasp the significance of these curves in showing the enormous increase in the vibrational energy necessary for the patient to just hear compared with a person possessing normal hearing.

To show the ability of the person to check the data the dotted curves, taken six months later, are shown in figures 3 and 5. The curves not only check the earlier data but also show in the best possible manner that there has been no change to speak of during the six months' period.

Several curves of this type have been obtained for different patients, but with the depressions showing up at different frequencies. We may ask how is it possible for so serious a defect to affect the hearing in only a single region of this sort? Without going into the anatomical evidence regarding the structure of the ear, we may say that physically the affected parts in these cases are evidently very loosely coupled (speaking mechanically) to the rest of the ear which functions normally out side of this region. Such would not be the case for middle ear trouble as shown by figures 2 and 4. The limit of audibility for this patient, who is 25 years old, is about 12000 cycles which is lower than we should expect for a person of this age. So that, the whole region above 3000 cycles (roughly speaking) is probably affected. From arguments to be presented elsewhere[2] it may be that this lesion is one affecting the transverse fibres of the basilar membrane. The data at hand seem to indicate that these fibres function in somewhat the same way as postulated by Helmholtz and others (not however as pitch "determinators.") Hence, possibly the fibres above 3000 cycles are prevented from performing their proper function while those below this frequency are not yet affected.

Auditory Nerve Endings.—In concluding this brief account it is worth while to show how the data suggest the part played by the nerve endings, not as organs for changing mechanical energy into auditory nerve energy, but as organs which respond only to a very narrow range of frequencies where the pitch discrimination is good and which respond to wider ranges where the discrimination is not so good. Some curves were taken which showed that within a range of one hundred cycles the hearing decreased suddenly so that the patients were unable to hear even though the vibrational energy of the receiver diaphragm was made 10,000 times as great as was necessary for them to hear at the lower frequency and perhaps 10^{10} as large as was required for a normal ear to hear at the frequency in question. With this amount of energy the region of the internal ear corresponding to the lower frequency was certainly vibrating with an amplitude considerably in excess of that necessary for audibility. Yet, no tone was heard. In other words, apparently it would be necessary to stimulate the nerve corresponding to the higher frequency if the tone of the higher pitch is to be heard. So that, apparently a nerve will respond only to one frequency or a very narrow band of frequencies. The actual pitch determination, then, resides in the nerve endings which appear to have this highly selective action. It seems probable, then, that this work is leading into fields which can be explored by precise physical measurements and fields which up to the present time have not been covered satisfactorily by other means of investigation.

The writer is greatly indebted to Dr. J. Gordon Wilson, Head of the Department of Otology Northwestern Medical School, with whom this co-operative research has been done and to whom equal credit for the success of the work is due. The writer is also indebted to Professors Millikan and Lunn of the Department of Physics of the University of Chicago for their helpful suggestions and enthusiastic interest during all the stages of development up to the present time.

[1] John P. Minton and J. Gordon Wilson, "Sensitivity of Normal and Defective Ears for Tones of Various Frequencies," *Proc. Inst. of Medicine, Chicago,* 1921.

[2] John P. Minton, "Physical Characteristics of the Ear," in preparation for the *Physical Review.*

NON-DISJUNCTION AND THE CHROMOSOME RELATIONSHIPS OF DROSOPHILA WILLISTONI

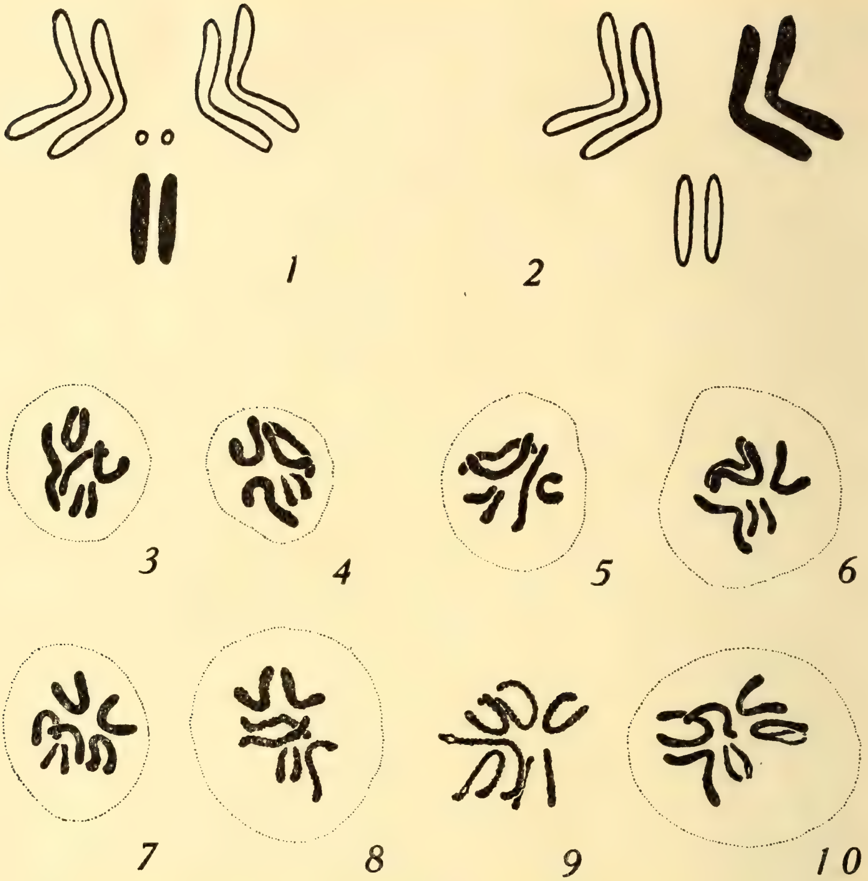
BY REBECCA C. LANCEFIELD AND CHARLES W. METZ

STATION FOR EXPERIMENTAL EVOLUTION, COLD SPRING HARBOR, N. Y.

Communicated by C. B. Davenport, May 5, 1921

Of the eleven types of chromosome groups found in the genus Drosophila (Metz '16)[7] that designated type A is the most widespread—occurring in 13 of the 29 species that were studied. This type is represented in figure 1. It consists of one pair of rod-like chromosomes, two pairs of long, V-shaped chromosomes and (usually) one pair of very small spherical "m" chromosomes.[1] The constancy of this type among the 13 species is such as to suggest a genetic homology of the respective pairs of chromosomes throughout. On the other hand the species themselves are scattered more or less at random through the genus and do not constitute a restricted taxonomic group. This suggested the desirability of a comparison of the genetic constitution of the chromosomes in two or more of these species.

Since D. melanogaster was already well known genetically and cytologically, and thus afforded a convenient basis of comparison, we undertook a study of D. willistoni,[2] a species resembling melanogaster in appearance and in chromosome constitution. In the course of this study we have found, by an examination of non-disjunctional flies, that the chromosomal resemblance between willistoni and melanogaster is misleading, for the sex chromosome pair of willistoni does not correspond morphologically to that of melanogaster but corresponds, rather, to one of the autosome pairs of this species. This would indicate that corresponding chromosomes in the two species are not all strictly homologous. The evidence for this conclusion together with a brief discussion is given below.



Figures 1 and 2 are diagrams; figures 3–10 are camera lucida drawings from fixed and sectioned material.[4] Figure 1. Chromosome group of Drosophila melanogaster female; X chromosomes represented in solid black. Figure 2. Chromosomes group of Drosophila willistoni; X chromosomes in solid black. Figures 3 and 4. Normal female groups of willistoni. Figures 5 and 6. Normal male groups of willistoni. Figures 7–10. XXY groups from non-disjunctional females of willistoni.

Cytological Data.—The normal female chromosome group of Drosophila melanogaster is represented diagrammatically in figure 1 and that of D. willistoni in figures 2–4. In melanogaster, as shown by Bridges[6] ('16) in his work on non-disjunction, the rod-like chromosomes are the X chromosomes.[3] These are represented in solid black in the diagrams. In the male of melanogaster this pair is asymmetrical, X being rod-like and Y J-shaped. In the male of willistoni no asymmetry is evident in any pair (figs. 5, 6). Otherwise the chromosomes of willistoni correspond, member for member, with those of melanogaster, except for the possible absence of the minute, spherical pair.

The lack of inequality in the sex chromosomes of the willistoni male necessitated obtaining non-disjunctional flies with three sex chromosomes in order to distinguish sex chromosomes from autosomes. As shown by the genetic data given below these non-disjunctional flies were females having one Y and two X chromosomes. Four chromosome groups from such females are shown in figures 7–10. In each of these there is an extra chromosome, and it is clearly a large and V-shaped one. Since this must be a sex chromosome it proves that the sex chromosome pair in willistoni is one of the long, V-shaped pairs and not the rod-like pair as would have been expected from analogy with melanogaster. We have obtained numerous clear cut figures both of the normal group in each sex and of the non-disjunctional group, and are confident that the evidence is entirely conclusive on this point.

Genetic Data.—In obtaining and identifying the non-disjunctional flies we have followed the procedure used by Bridges ('16) in the case of D. melanogaster. The following summarized account shows the resemblance between the results in the two species and indicates the source of the XXY chromosome groups described above.

Primary non-disjunction appears to be very rare in willistoni and no cases (which could be checked) were observed in our regular experiments. In approximately 150 cultures made especially for this purpose only two cases were detected. In each of these the exceptional fly was a female. The first came from a cross of a rough eyed female by an orange, small-bristle male, both from stock cultures. (Rough, orange, and small-bristle are all inherited as sex-linked recessives.) This mating gave 132 normal daughters, 111 rough eyed sons, and 1 rough eyed (exceptional) daughter (Culture W 1715). Since this rough daughter had already mated with a rough brother, it was not possible to detect "secondary" exceptions produced by her. Six of her rough daughters, however, produced 11 exceptional sons and daughters among 960 significant flies, or 1.14 percent of exceptions. In succeeding generations about the same ration was maintained, as is shown in table 1. For the sake of convenience, this strain has been designated as "line A."

The second primary exception appeared in an entirely unrelated stock, involving the characters two-bristle, short-3, and rough. It gave rise to a second strain known as "line B." Table 1 gives a summary of the breeding tests with this line, which gave results essentially similar to those of line A.

In addition to the ordinary cases of "reductional" non-disjunction we have also detected two cases of "equational" non-disjunction. This type was first described by Bridges ('16) who reported 18 "equational exceptions," i.e., daughters homozygous with respect to a recessive character for which the mother was only heterozygous. In most of his cases, he was able to demonstrate that crossing over had taken place in one chromosome and not in the other, while in two cases he showed that two crossover chromosomes were present in the exceptional daughter. He explained these facts as due to non-disjunction occurring at the equational division, and believed that they involved crossing over at the "four strand stage" in the primary oocyte.

Of the two certain cases of "equational" non-disjunction in willistoni, one gave a result like that just mentioned. The exceptional fly in this case came from an XXY female (W 1955) which had stubby, orange, small-bristle in one X chromosome, and orange, rough, short in the other. This female was crossed to a two-bristle, short-3 male and produced the following offspring: 135 daughters and 119 sons of the regular classes, 2 two-bristle, short-3 sons (secondary exceptions of the ordinary "reductional" type), and 1 stubby, orange, small-bristle daughter—an "equational" exception. This last female produced secondary exceptions as expected, but she also carried a sex-linked lethal so that her only surviving sons were stubby, orange, small-bristle. Her exceptional daughters, being of the same constitution, also gave this same lethal ratio, but breeding tests from her regular daughters revealed her constitution. By such means it was proved that she had one non-crossover chromosome carrying stubby, orange, and small-bristle, and one crossover chromosome carrying stubby, orange, small-bristle, rough and short, and the new lethal. A non-disjunctional strain was established from this female known as line D (table 1).

TABLE 1

SUMMARY OF BREEDING TESTS

LINE	TOTAL FLIES	EXCEPTIONS	PERCENTAGE OF EXCEPTIONS
A	6671	117	1.8
B	1131	13	1.1
C	701	3	0.4
D	497	22	4.4
Total	9000	155	1.7

The other equational exception is not of special interest since both of her chromosomes were non-crossover chromosomes, as far as the region that could be followed was concerned. It is not certain whether her mother, a descendant of line B, carried a Y chromosome. She gave rise to line C (table 1).

Additional possible cases of equational exceptions have been found but they could not be positively identified as such.

To secure XXY chromosome groups for cytological study the daughter of exceptional females from line A (W 1894b) were used. Theoretically half of these daughters should be of the desired constitution. No attempt was made to determine whether or not this 1:1 ration was actually realized, but XXY individuals were found without difficulty in the material put up for cytological study.

Discussion.—The evidence is not yet sufficient to indicate the exact relationship between the chromosomes of the two species considered here, but it does indicate that either the chromosomal resemblances are merely superficial or that the sex determining element (gene? or genes?) has been transferred from one chromosome pair to another. A comparison of the sex-linked mutant characters in the two species ought to throw some light on this question. It has not done so up to the present, however, for although we have obtained 27 such characters in willistoni they show so little resemblance to any in melanogaster (either sex-linked or non sex-linked) that they give no clue to chromosomal relationships.

The observed frequency of secondary non-disjunction in willistoni (average 1.7%) was less[5] than that found by Bridges in melanogaster (4.3%). There is no indication at present as to why this should be the case unless the size of the sex chromosomes be considered a factor.

[1] The "m" chromosomes are often difficult to detect. They may be lacking entirely in willistoni.

[2] D. willistoni Sturtevant (D. pallida Williston).

[3] Dr. Bridges kindly informs us that he has subsequently verified this conclusion.

[4] We are indebted to Dr. José Nonidez for making the drawings for figures 3–10.

[5] Line D may possibly be an exception but the small numbers make this doubtful.

[6] Bridges, C. B., *Genetics*, **1**, 1916 (16–52, 107–163).

[7] Metz, C. W., *J. Exp. Zool.*, **21**, 1916 (213–276).

AN APPARATUS FOR DETERMINATION OF THE GASES IN BLOOD AND OTHER SOLUTIONS

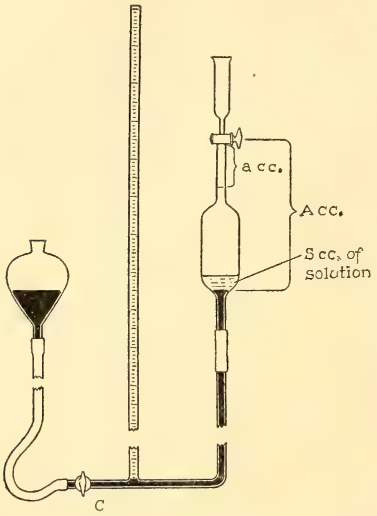
By Donald D. Van Slyke

Hospital of the Rockefeller Institute, New York

Communicated July 5, 1921

The apparatus consists of a pipette with the upper stem closed by a stopcock, the lower connected with a glass tube. The latter descends 800 mm., then turns at a right angle to connect with a levelling bulb and a mercury manometer open at the upper end. The pipette is calibrated at two points to hold *a* and *A* cc., respectively, as shown in the figure.

For an analysis the pipette is filled with mercury. The solution to be analyzed, followed by the reagents to free the gases (e.g., acid for CO_2 in carbonates) is admitted with slight negative pressure through the upper cock, displacing mercury in the pipette. A Toricellian vacuum is created by lowering the levelling bulb, and the meniscus of the mercury in the pipette is allowed to fall to the mark indicating *A* cc., as shown in the figure. Cock *c* is closed, and the pipette is shaken for the time required to establish equilibrium of the gases between the solution and the free space above it. One to two minutes usually suffice. Mercury is then readmitted at cock *c* until the gas volume in the pipette is reduced to *a* cc. Cock *c* is closed and the height of the mercury column in the manometer is read (*m* mm.). The zero point is then determined after expelling the gases from the apparatus, or after absorbing one or more of them by introduction of small, measured volumes of gas-free absorbent solutions (KOH for CO_2, pyrogallol for O_2) through the upper cock under slight negative pressure. After the gas has been removed the pressure is lowered until the free space above the solution is again *a* cc., and the zero point for the determination is read on the manometer (*n* mm.).



The volume, *V*, of gas reduced to 0°, 760 mm., contained in the solution analyzed is calculated as:

$$V = a\frac{m-n}{760}\left(\frac{273}{T} + \frac{S\alpha}{A-S}\right)$$

T = absolute temperature, S = volume of water solution in the apparatus, α = solubility coefficient of the gas in the solution (the cc. of gas, reduced to 0°, 760 mm., dissolved by 1 cc. of solution in equilibrium with the gas at 760 mm. tension).

The term $\frac{S\alpha}{A-S}$, which corrects for the portion of gas remaining in solution when equilibrium is reached, may be negligible for the less soluble gases, such as oxygen and nitrogen, but not for CO_2. The term is derived as follows. If V_S = volume of gas (measured at 0°, 760 mm.) held in solution by the S cc. of solution, and p = partial pressure of the gas

$$V_S = S \times \alpha \times \frac{p}{760}$$

$$p = (m-n) \times \frac{a}{50-S}$$

$$V_S = a\frac{m-n}{760} \times \frac{S\alpha}{A-S}$$

In case x cc. of absorbent solutions are introduced, a correction to n is necessary. It is ascertained by determining m for S and for $S + x$ cc. of water, respectively, the dissolved gases being removed by expulsion.

The solubility of CO_2 also makes an empirical correction necessary for reabsorption of the gas while the volume is undergoing reduction from $50 - S$ cc. to a cc. In our apparatus, where $S = 50$ cc., $a = 2$ cc., the factor is approximately 1.020, the reabsorbed CO_2 being 2 per cent of the total. For the less soluble gases reabsorption may be kept negligible.

No correction for vapor tension is required, since it is practically the same at the reading of both m and n.

For a given gas volume the value of the pressure change $(m - n)$ is inversely proportional to that of a. a may accordingly be so chosen that for the gas volumes obtained the percentage error in measuring a cc. of gas will be of the same order of magnitude as that in measuring the accompanying $(m - n)$ mm. of pressure change. The total volume A of the pipette chamber is a matter of convenience, but it is desirable to have it so large that the greater part of the dissolved gases shall be extracted. For analysis of 1 cc. of blood convenient magnitudes are $A = 50$, $a = 2$, $S = 2.5$. At 20° 1 volume per cent of gas under these conditions gives a reading of $m - n = 3.9$ mm., so that if $m - n$ can be determined within 0.4 mm. the error is 0.1 cc. of gas per 100 cc. of blood.

ADAPTIVE RADIATION AND CLASSIFICATION OF THE PROBOSCIDEA[1]

By HENRY FAIRFIELD OSBORN

AMERICAN MUSEUM OF NATURAL HISTORY, NEW YORK CITY

Read before the Academy, April 26, 1921

In 1900 the author predicted that the source of the mammalian order of the Proboscidea would probably be discovered in Africa. In 1901 Beadnell and Andrews revealed, through the Geological Survey of Egypt, the rich fauna of the Fayûm, southwest of Cairo, in which were found the remains of three proboscidean genera, named by Andrews *Mœritherium, Palæomastodon, Phiomia,* confirmed by subsequent exploration and research to be the oldest proboscideans thus far known. Animals similar to *Mœritherium* and *Phiomia* have since been reported by Pilgrim in southern Asia. These animals are now found to belong respectively to three distinct lines of the Proboscidea, namely, the moeritheres, the true mastodonts, the long-jawed bunomastodonts, as indicated in black on the accompanying diagram. They point, however, to a long antecedent origin and radiation. This is part of the evidence for an ancient adaptive radiation process by which it now appears that the proboscideans, like other hoofed mammals, were broken up into several great primary stocks way back in Eocene times, namely:

An *amphibious stock,* adapted to rivers and swamps, of limited migration.

A *mastodont stock,* adapted to forests and savannas, of wide migration.

An *elephant stock,* adapted to grassy plains, savannas, and steppes, of wide migration.

These primary stocks gave off from two to six branches each, so that the Proboscidea as a whole are not two branched (i.e., mastodonts and elephants), as formerly supposed, but many branched or *polyphyletic.* The forest and savanna browsers and the grazers of the plains and steppes were the long distance travelers and from an African or Asiatic center in Eocene times they reached in the Middle and Upper Miocene all the continents of the world except Australia, while the amphibious forms remained in Africa and southern Eurasia. Certain of these branches, like the true mastodons, are of very great geologic antiquity. Intelligent, independent, well defended, resourceful, adaptive, we find eleven very distinct branches of proboscideans persisting into Upper Pliocene times, five of the least hardy of which became extinct during the colder conditions of the Lower Pleistocene.

The known lines of evolution are shaded on the accompanying diagram; the unknown are left in white. The adaptive radiation may be expressed in a formal classification as follows:

Amphibious and swamp-living stock

I. MOERITHERIOIDEA (Mœritheres)

1. Mœritheriini,[2] amphibious or swamp-living forms known in the Upper Oligocene of Africa.

II. DINOTHERIOIDEA (Dinotheres)

2. Dinotheriini,[3] large amphibious forms frequenting the rivers of southern Eurasia throughout the Miocene to the close of the Pliocene.

Forest and savanna grazers

III. MASTODONTOIDEA (Mastodonts and Bunomastodonts)

MASTODONTIDAE OR "true mastodonts," including the subfamily

3. Mastodontinæ, springing from *Palæomastodon* of the Oligocene of North Africa, and terminating with *Mastodon americanus* of the Pleistocene forests of North America; grinders lophodont, lacking trefoils.

4. Serridentinæ,[4] first known in the Middle Miocene of France and Switzerland, spreading over into India and North America; lacking the trefoils.

BUNOMASTODONTIDAE, the bunomastodonts, springing from forms similar to the *Phiomia* of North Africa and separating into four main divisions:

5. Notorostrinæ, a special branch entering the Andean region of South America and spreading over the South American continent, distinguished by the loss of the lower tusks and the abbreviation of the jaw.

6. Longirostrinæ, typical long-jawed bunomastodonts arising in North Africa (*Phiomia*), spreading all over southern Europe, Asia, and North America, and giving off:

7. Rhynchorostrinæ, beaked bunomastodonts, known only in the southern United States and northern Mexico, with powerful downturned upper and lower tusks.

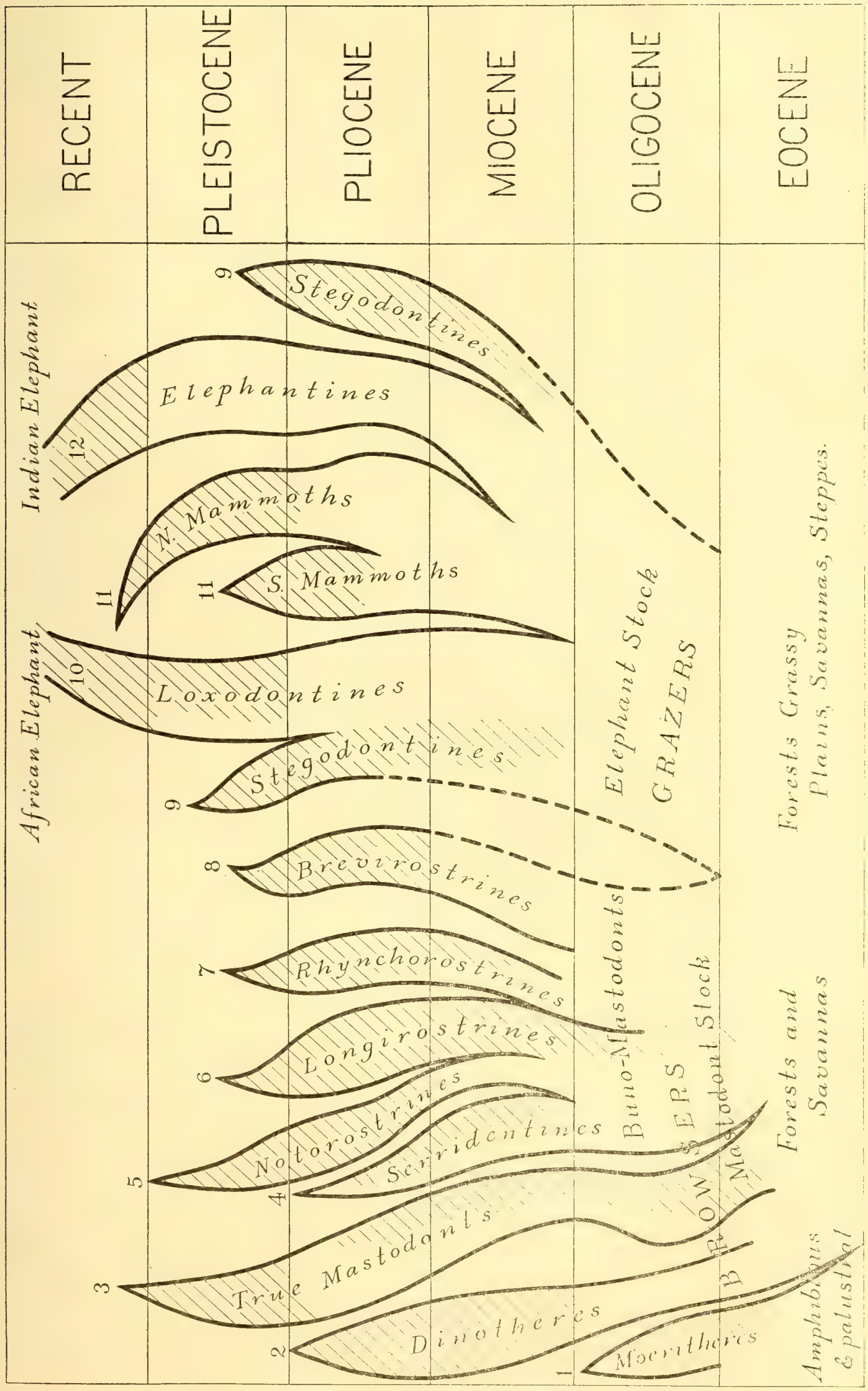

RECENT
PLEISTOCENE
PLIOCENE
MIOCENE
OLIGOCENE
EOCENE
Indian Elephant
African Elephant
12
Elephantines
9
Stegodontines
11
N. Mammoths
11
S. Mammoths
10
Loxodontines
9
Stegodontines
8
Brevirostrines
7
Rhynchorostrines
6
Longirostrines
5
Notorostrines
4
Serridentines
3
True Mastodonts
2
Dinotheres
1
Moeritheres
Elephant Stock
GRAZERS
Buno-Mastodonts
BROWSERS
Mastodont Stock
Forests Grassy
Plains, Savannas, Steppes.
Forests and
Savannas
Amphibious
& palustral

FIG. 1. Diagram showing the present theory as to the adaptive radiation of the Proboscidea. June, 1921.

8. Brevirostrinæ, short-jawed bunomastodonts, which imitate both the true mastodonts and the elephants in the abbreviation of the lower jaw and the early loss of the inferior tusks. These animals wandered all over Europe, Asia, and western North America.

IV. ELEPHANTOIDEA (the Elephant stock)

9. Stegodontinæ, the original members of which were doubtless ancestral to all the higher elephants, persist as an independent branch into the Lower Pleistocene of eastern Asia.
10. Loxodontinæ, embracing the great African division of the elephants beginning with *Loxodonta antiqua* of the Upper Pliocene, which wandered all over southern Eurasia and radiated widely over Africa.
11. Mammontinæ, including (*a*) the Southern Mammoths (*Elephas planifrons* of India and *E. Meridionalis* of Europe), from which is derived *E. imperator* of North America, and (*b*) the Northern Mammoths, which probably include *E. columbi* and the widespread *E. primigenius of the* northern steppes; tetradactyl pes.
12. Elephantinæ, the true elephants (*E. indicus of* India), which do not appear until the Upper Pleistocene; pentadactyl pes.

This twelve-fold branching of the proboscideans is similar to the adaptive radiation which the author has traced in the evolution of the horses, of the rhinoceroses, and of the titanotheres, carrying the fundamental lines of separation back to the Middle Miocene as the most recent date, and to the Middle or Lower Eocene as the most remote date.

It will be observed from the diagram that the shaded areas represent those phyla of which remains have been discovered. The large unshaded area includes the entire Oligocene, Miocenes, and Lower and Middle Pliocene history of the Elephantidæ which is still unknown but which is likely to be revealed at any time by discoveries both in Africa and in central Asia. A very striking fact is that the geologically earliest known member of the Elephantoidea is the *Elephas planifrons* of the Upper Pliocene of India, the apparent ancestor of the mammoths.

[1] The first paper in this series is entitled, "A Long-jawed Mastodon Skeleton from South Dakota and Phylogeny of the Proboscidea," *Bull. Geol. Soc. Amer.*, March, 1918; the second paper, "Evolution, Phylogeny, and Classification of the Proboscidea," *Amer. Mus. Novitates*, No. 1, January 31, 1921 (Osborn, 1921. 514); the third paper, "First Appearance of the True Mastodon in America," *Amer. Mus. Novitates*, No. 10, June 15, 1921; the fourth paper appears in the *Bulletin* of the Geological Society of America, under the title, "Evolution, Phylogeny, and Classification of the Mastodontoidea;" the present is the fifth paper. The Iconographic Type Revision will form one of the Memoirs of the American Museum of Natural History.

[2] Herluf Winge, 1906, p. 172.

[3] *Ibid.*

[4] It is a question whether this subfamily is nearest the Mastodontidæ, with which its members are generally placed by European palæontologists.

A CASE OF REARRANGEMENT OF GENES IN DROSOPHILA[1]

BY A. H. STURTEVANT

COLUMBIA UNIVERSITY, NEW YORK CITY

Communicated by T. H. Morgan, May 28, 1921

Seven mutant genes of Drosophila simulans have been shown to be allelomorphic to previously known mutant genes of D. melanogaster.[2] Five of these lie in the X-chromosome, and a study of their linkage relations was shown to indicate that the sequence of the five loci concerned is the same in both species, and that the percentages of crossing over in comparable regions, while not indentical, is still not very different. The other two allelomorphic mutant genes, scarlet and peach, lie about 3 units apart in the third chromosome of melanogaster; in simulans they lie in the same chromosome (which is thus identified as the third one), but they were found to be at least 45 units apart.

More recently two more mutant genes of simulans that lie in the third chromosome have been studied. One of these, dachs,[3] lies to the left of scarlet; the other, deltoid,[4] lies between scarlet and peach. The latter, since it makes possible the detection of a portion of the double crossovers, has resulted in a more accurate determination of the scarlet peach distance. The map based on the linkage relations of these four loci (not corrected for unobservable double crossing over) is shown in figure 1.

Both of the new mutant types, dachs and deltoid, resemble previously known mutant types in D. melanogaster. Dachs in melanogaster lies in the second chromosome; it is accordingly not surprising that tests have shown it not to be allelomorphic to dachs simulans. Deltoid resembles delta melanogaster. Since both genes are dominant, the usual test of allelomorphism could not be applied; but each has also a recessive lethal effect, and crosses of delta melanogaster by deltoid simulans have shown that the hybrids that receive both mutant genes do not develop. It follows that the two genes are allelomorphic. The map of the melanogaster third chromosome, including the known ends and the three loci occupied by parallel mutations, is shown in figure 2.[5]

A comparison of figures 1 and 2 shows that *the three identical loci are not in the same sequence in the two species.*

Mr. D. E. Lancefield has obtained evidence suggesting a similar rearrangement of genes in the X-chromosome of D. obscura. His results were obtained before those here reported, but are not yet published. Since D. obscura has not yet been crossed with any other species, the evidence for identity of loci is not conclusive in this case.

The only analogous case so far reported appears to be that briefly described by Bridges[6] under the name of "vermilion duplication." In this case a section from near the middle of an X-chromosome of D. melanogaster appears to have broken loose and attached itself to the left-hand end of a normal X-chromosome. Weinstein[7] has shown that such an occurrence might lead to a change of sequence of identical loci such as is here reported. If we suppose that the simulans third chromosome was originally constituted as is that of melanogaster, the situation as we now find it may be supposed to have arisen as follows: A section, including the peach locus, broke loose and attached itself near the right-hand end of a normal third chromosome. After this condition had become established the peach locus near the middle of the chromosome mutated or became "deficient," so that in effect the peach locus was moved to the right end of the chromosome. Such an interpretation will account for the observed facts.[8]

There is, however, another possible method whereby the same result might be supposed to have been brought about, viz., by the simple inversion of a section of a normal chromosome. Such an accident seems not unlikely to occur at the stage of crossing over. If we suppose a chromosome to occasionally have a "buckle" at a crossing over point, it is conceivable that crossing over might be followed by fusion of the broken ends in such a way as to bring about an inversion of a section of chromosome.

Either of the two suppositions discussed will account for the observed results, but they should lead to different relations for other loci in the same chromosome; it is hoped that further work will lead to the discoveryof additional parallel mutations, so that the maps may be studied in more detail.

If an inversion of the kind suggested above occurred within a species, then individuals bearing one normal chromosome and one chromosome with an inverted section would probably show no crossing over in the region in question, since it seems probable that synapsis in this region would be abnormal or absent. It would also be not surprising if crossing over in adjacent regions was decreased. But individuals homozygous for the inverted section would be expected to show free crossing over again, since there should now be no difficulty at synapsis.

The relations indicated are those that have actually been found in the cases of the two "crossover genes" in melanogaster known as C_{III}[9] and C_{IIr}.[10] These "genes" both cause, in individuals heterozygous for them, the disappearance of crossing over in the immediate regions where the "genes" themselves lie, and a considerable reduction of crossing over in neighboring regions. In individuals homozygous for either of these "genes," however, the percentage of crossing over rises to (or beyond) that found in "normal" individuals. Experiments are now under way in an attempt to determine if these "genes" are really simply inverted chromosome sections, but it will probably be a long task to definitely settle the matter.

The demonstration of a change in sequence of identical loci that is here reported makes the identification of parallel mutations in species that cannot be crossed even more difficult than it has previously seemed; for identity of sequence in a group of identical loci now appears not to be necessarily expected.

0.0	dachs
14.2	scarlet
39.0	deltoid
76.5	peach

FIGURE 1

0.0	roughoid
41.5	scarlet
44.5	peach
63.5	delta
101.0	minute-23

FIGURE 2

[1] Contribution from the Carnegie Institution of Washington.

[2] Sturtevant, A. H., *Genetics*, **6**, 1921 (63, 179).

[3] Discovered by Prof. T. H. Morgan.

[4] Discovered by Dr. C. B. Bridges.

[5] This map is based on the more extensive one published by Bridges in these PROCEEDINGS.

[6] Bridges, C. B., *J. Gen. Physiol.*, **1**, 1919 (645).

[7] Weinstein, A., these PROCEEDINGS, **6**, 1921 (625).

[8] It is, of course, possible to invert this interpretation by supposing the simulans situation to be the original one.

[9] Muller, H. J., *Amer. Nat.*, **50**, 1916 (103, 284, 350, 421), and Sturtevant, A. H., *Carnegie Inst. Wash. Publ.*, No. 278, 1919 (305).

[10] Sturtevant, A. H., these PROCEEDINGS, **3**, 1917 (555), and *Carnegie Inst. Wash. Publ.*, No. 278, 1919 (305).

A REMEASUREMENT OF THE RADIATION CONSTANT, h, BY MEANS OF X-RAYS

BY WILLIAM DUANE, H. H. PALMER AND CHI-SUN YEH

JEFFERSON PHYSICAL LABORATORY, HARVARD UNIVERSITY

Communicated July 6, 1921

Since the discovery of the fact[1] that the continuous X-ray spectrum has a short wave-length limit, which obeys the quantum law, a number of experimentors have used this phenomenon to determine the value of h.[2] In its application to X-rays the quantum law may be expressed by the equation

$$Ve = h\nu, \qquad (1)$$

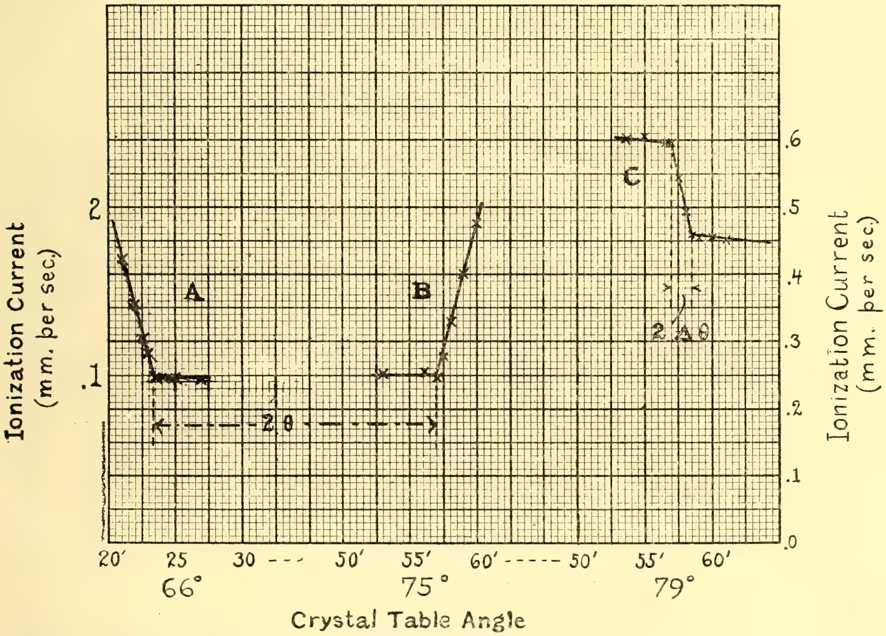
where V represents the maximum difference of potential in the X-ray tube through which the electrons fall, e, the charge carried by each electron, ν, the frequency of vibration corresponding to the short wave-length limit of the spectrum, and h, Planck's action constant. Evidently a measurement of V and ν gives us the ratio of h to e, and from this we get h, if we suppose e to be given by other experiments. Blake and Duane[3] made the most accurate measurement of h in our laboratory. Using Millikan's value of $e = 4.774 \times 10^{-10}$ they obtained the value $h = 6.555 \times 10^{-27}$.

The object of the research reported in this note is to increase the accuracy of the measurement of h. We use a new and somewhat improved spectrometer and a new calcite crystal. The X-ray tube contains a tungsten target and a Coolidge cathode. A side arm attached to the tube extends out toward the spectrometer and carries at its outer end a thin mica window. The increased intensity of the X-rays coming through this window enables us to use a narrower spectrometer slit, which reduces the correction that must be made for the slit's width.

As in the previous researches the high tension storage battery supplies the current through the X-ray tube. In the present research we have greatly increased the accuracy of the measurement of the difference of potential applied to the tube. Whereas in the previous researches this difference of potential was compared with the electromotive force of a standard cell through the calibration of several intermediate instruments (an electrostatic voltmeter, an ammeter, and a potentiometer), we now compare the difference of potential directly with the electromotive force of a standard cell by means of the simple potentiometer method. In this way we eliminate the errors in the calibrations of the various instruments. The main circuit in the simple potentiometer consists of a large number of coils of manganin wire having a total resistance of over six million ohms. We use the same precautions in insulating the various circuits as one employs in making measurements of ionization currents. The ratio of the resistances of two sections of the main circuit gives us directly the ratio of the difference of potential to the electromotive force of the standard cell. As the ratio of two resistances can be measured with extreme accuracy, we think that we know the difference of potential applied to our X-ray tube with about the accuracy with which the electromotive force of the standard cell has been determined. We use two unsaturated Weston standard cells, each of which has been tested at the Bureau of Standards, and we compare them with each other from time to time. The certificates from the Bureau give the electromotive forces of the cells to within one part in ten thousand.

The actual drop in potential in an X-ray tube through which the electrons fall differs from the drop in potential measured by the potentiometer by a small amount due to the Volta effect, the current that heats the coil of wire in the Coolidge cathode and the high temperature of the wire. In our experiments the circuits are so connected and the voltage applied to the tube is so high that we may neglect the corrections due to these effects.

In making a measurement of h, one experimenter observes the galvanometer attached to the potentiometer, and by varying a resistance in series with the X-ray tube keeps the difference of potential applied to it constant during the experiment. A second observer measures the current in the spectrometer's ionization chamber. A series of readings is taken on the two sides of the spectrometer's zero near the points at which the continuous X-ray spectrum vanishes. Curves A and B in the figure represent the ionization currents as functions of the angles that fix the positions of the reflecting crystal in one of the experiments. The horizontal portions of the curves correspond to the currents due to natural leak and to stray radiation.



The inclined portions represent the increase in these currents due to the continuous X-ray spectrum. The readings corresponding to the short wave-length limit of the spectrum can be determined to within a few seconds of arc. As indicated in the figure, the difference between these readings on the two sides of the zero gives us twice the glancing angle θ, which must be substituted in the equation

$$\lambda = 2d \sin \theta = 6.056 \times \sin \theta \times 10^{-8} \text{cm}. \qquad (2)$$

in order to calculate the shortest wave-length λ, in the continuous X-ray spectrum.

A small correction has to be added to the observed value of θ, due to the fact that the source of rays and the slit of the spectrometer are not mathematical lines. The correction for the breadths of the source and slit we determine in two ways. Firstly, we estimate the apparent breadth of the focal spot as seen from the spectrometer's slit, and then measure the width of the slit by the method described on p. 630 of the paper by Blake and Duane referred to above.[3] Secondly, we measure the breadth of the drop in the ionization curve due to the *K* critical ionization of bromine. The ionization chamber containes ethylbromide. Curve *C* in the figure represents this drop in the curve in one of the experiments. The breadth of this drop corresponds within the limits of error of the measurements to the correction to be added to the double angle 2θ, as determined by the first method. The correction is small. It amounts to less than one part in three hundred.

Incidentally, the measurements we have made of the critical ionization of bromine gives a very accurate measurement of its critical ionization wave-length. As an average we obtained the value

$$\lambda = (.9180 \pm .0002) \times 10^{-8} \text{ cm.},$$

assuming that the grating constant of calcite is 6.056×10^{-8}cm.

We tried to obtain an estimate of the correction for the breadth of the source and the slit by measuring the breadth of a peak on the ionization curve corresponding to a line in the characteristic emission spectrum of the tungsten target. In every case examined, however, the breadth of the peak appeared to be slightly broader than what the measured breadth of the source and slit indicated it should be. This means that the corresponding emission lines have certain finite, intrinsic breadths of their own. If the *K* critical ionization of bromine has such an intrinsic breadth it is too small to be detected in these experiments.

Since from the wave equation,

$$\lambda \times \nu = c = 2.9986 \times 10^{10}$$

where c is the velocity of light, we can calculate immediately the maximum frequency ν in the continuous spectrum. This, together with the difference of potential, V, substituted in equation 1, gives us the ratio of h to e.

The data obtained in four complete measurements of h appear in table 1. Column 2 contains the uncorrected values of θ, column 3 the correction for the breadths of the source and of the slit, and column 4, the corrected values of θ. Column 5 contains the values of the product $V \sin\theta$ (which is what we really measure in our experiments). This product depends to a slight extent upon the temperature. The temperature in these experiments

TABLE 1

GLANCING ANGLES AND RADIATION CONSTANT

DATE	θ(UNCORRECTED)	$\Delta\theta$	θ(CORRECTED)	$V \sin\theta$	h
March 15	4°–46′–43″	47″	4°–47′–30″	2039.2	6.5539×10^{-27}
March 21	4°–46′–53″	43″	4°–47′–36″	2039.9	6.5561 "
March 30	4°–46′–53″	51″	4°–47′–44″	2041.0	6.5594 "
April 5	4°–46′–48″	45″	4°–47′–33″	2039.6	6.5552 "
			Mean	2040.0	6.5562×10^{-27}

is about 20° Centigrade. The value of V as measured by the potentiometer is 24,413 volts in each case. We estimate that the error of precision in the product $V \sin \theta$ amounts to about one part in two thousand.

Column 6 contains the values of h calculated from the values of $V \sin \theta$. Since several constants occur in these calculations (the velocity of light, the grating constant of calcite and the charge on the electron), the values of h are not quite as accurate as those of $V \sin \theta$. We estimate the probable error in h to be about fifteen parts in ten thousand. This gives us an average

$$h = (6.556 \pm .009) \times 10^{-27}.$$

This value of h agrees with that previously published by Blake and Duane.[3] It is, however, a fraction of one per cent larger than the value recently obtained by E. Wagner in a careful series of measurements.

In the experiments described above the X-rays left the target in a direction at right angles to the line of motion of the cathode particles. An interesting question has been raised recently as to whether the limit of the continuous spectrum would be altered if the rays came off at some other angle.[4] To test this point with the accurate method of measuring the voltage which we now have, we made a series of experiments with an ordinary Coolidge tube (tungsten target) placed so that the X-rays that passed through the spectrometer's slit left the target at an acute angle of about 45° to the direction of the cathode stream. The results of this series of measurements appear in table 2. As in the previous experiments the voltage applied to the tube amounted to 24,413 volts. The X-ray tube had no thin mica window, so that the accuracy appears to be somewhat less than in the previous series of measurements. The value of $V \sin \theta$, however, does not differ from that for rays at right angles to the cathode stream. There does not appear to be a Doppler effect for the short wave-length limit of the spectrum that amounts to as much as one part in two thousand. This agrees with Wagner's results.[4]

TABLE 2

GLANCING ANGLES AND RADIATION CONSTANT

DATE	θ(UNCORRECTED)	$\Delta\theta$	θ(CORRECTED)	$V \sin \theta$	$h \times 10^{-27}$
April 6	4°–46′–35″	1′	4°–47′–35″	2039.8	6.556
April 6	4°–46′–33″	1′	4°–47′–33″	2039.6	6.555
April 12	4°–46′–53″	48″	4°–47′–41″	2040.7	6.558
April 27	4°–46′–43″	47″	4°–47′–30″	2039.3	6.554
				2039.9	6.556

A very much more detailed report of this research is being published in one of the Physical Journals.

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[1] Duane and Hunt, *Physic. Rev., Ithaca*, Aug., 1915, p. 166.

[2] See a report in the *Jahrbuch der Radioahctivität*, etc., for 1919 by E. Wagner, and

also a report on "Data Relating to X-Ray Spectra" by William Duane, published by the NATIONAL RESEARCH COUNCIL.

[3] Blake and Duane, *Physic. Rev.*, Dec., 1917, p. 624.

[4] See E. Wagner, *Jahrbuch der Radioahctivität*, 1919, also *Physik. Zeit.*, Nov., 1920, p. 621; and C. Zecker, *Ann. Physik. Leipzig*, Sept., 1920, p. 28, also a note by D. L. Webster, presented to the American Physical Society at the same meeting at which the authors presented a note on this research, April, 1921.

THE U-TUBE ABSOLUTE ELECTROMETER

By CARL BARUS

BROWN UNIVERSITY

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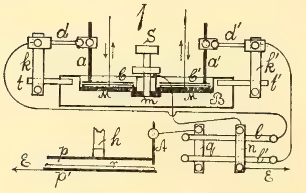
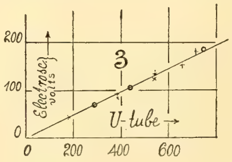
1. *Electrical Condenser.*—Adjusting the wide shank of the shallow U-tube heretofore described (these PROCEEDINGS, **7**, 1921, p. 71) with the top plates removed, so as to admit a metallic disc above the earthed mercury surface and parallel to it, the device becomes an absolute electrometer. The disc is perforated at the middle so that the component rays of the interferometer may reach the mercury. This instrument is chiefly useful in measuring electrostatic potentials. If p is the electric-pressure below the disc charged at potential difference V and h is the head of mercury resulting

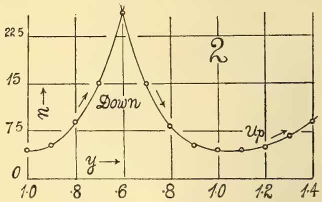
$$V = d\sqrt{8\pi p} = d\sqrt{8\pi h\rho g} = d\sqrt{4\pi\lambda\rho g.n}$$

where d is the distance between disc and mercury, ρ the density of mercury and λ the wave-length of light when n fringes correspond to V. Hence if $d = 1$ millimeter, $V = .315\sqrt{n}$ els. units; or $= 95\sqrt{n}$ volts.

2. *Improved Apparatus.*—The electrometer eventually took the form shown in figure 1 which gives the apparatus in connection with the electrophorus and a commutating key similar to Mascart's. To put the mercury M to earth, a steel screw, S, which also carries a flat clamp for fastening one end of the earthed wire, has been inserted. This screw, S, has the further important purpose of damping the oscillations of the mercury M or M', by adjustably closing the channel m. The deflections can thus be made quite dead beat, which is an advantage. To level the electrodes C, C' (using a small spirit level placed on them) each has connecting rod d, which carries a clamp at one end, allowing the rod a, a' to slide up and down, rotate







around a, a' and d, d' and admitting of small displacements along d. At the other end of each rod is a flat vertical plate which is received in a fissure at the top of the corresponding hard rubber post k, k' and clamped. This gives a horizontal axis at right angles to the preceding. The lower ends of the posts k, k' are suitably clamped to the tubes t, t', attached to the body B of the electrometer. Here further motion along the tubes t, t' and rotation around them is possible. In this way it is not difficult to place C, C' symmetrically above the mercury pools and parallel to their upper faces, for experimental purposes. It is not sufficient, however, if precision is required.

Figure 1 shows the Mascart key below on the right, which consists of the elastic brass strips l, l', the earthed cross bar n above them and the cross bar q below. Thus the whole U-tube is earthed when not in use. The bar q is connected by wires with the brush A of the electrophorus shown on the left, so that when l or l' are depressed into contact with q, C or C', respectively, receives a positive charge while the other electrode and the mercury is earthed. This affords a very satisfactory means of commutation; for since $\Delta V = C\sqrt{n} = C'\sqrt{n'}$, the electrodes are so adjusted that C and C' are nearly equal.

Tests were made with this apparatus and a known $\Delta V = 173$ volts. For example, the scale readings in the ocular on commutation were $x = 34$, $x' = 17$ so that $(x - x') = .7\ (n + n')$, as the fringe breadth was .7 scale parts. Thus $\Delta V = A\ \sqrt{24}$, or $A = 35.3$ volts per fringe, initially. With large fringes and under quiet surroundings 3 or 4 volts could have been detected.

The upper face of the electrophorus p is on a vertical micrometer screw, insulated by the hard rubber connector h. The distance apart of p, r and p' (to be denoted by d' and d'') or any change of this distance (Y) are thus closely measurable in turns (mm.) of the screw. In a dry room this apparatus retains its charge Q very well and a great variety of fields are producible.

3. *Equations.*—If we treat the case of the electrophorus as a closed cylindrical field of cross section A, and if V_0 is the potential of the charged hard rubber surface, we may write

$$Q' = \frac{AK'}{4\pi d'}(V' - V_0)$$

where Q', V' are the positive charge and potential in the top plate at a distance d' from the charged rubber surface at potential V_0 and K' the specific inductive capacity of the dielectric medium. A similar equation holds if Q'', V'' are the charge and potential of the lower plate at a distance d'' from V_0 with a layer of specific inductive capacity K'' between.

If the two plates are put in contact, $V' = V''$.

If the two plates thus charged are then insulated and the top plate is moved normally towards the lower, a distance of y, the equations reduce to

$$-\Delta V = 4\pi Q d'' y / A(K''d' + K'd'') = \text{const. } \sqrt{n},$$

ΔV being the potential difference thus produced and measured at the U-tube electrometer taken as small in capacity in comparison with the electrophorus.

$Q = Q' + Q''$. Hence, as a first approach, the y, n locus is a parabola. For instance in the following example the insulation loss amounted to not more than 2 fringes in 10 minutes at full charge. The pitch of the micrometer screw being .1 cm., the upper plate was conveniently discharged when $d' = 1$ cm. above the hard rubber surface. Large fringes (about 1.5 scale parts) were installed. The fringe displacements (n) observed on lowering and raising the plate are shown in figure 2. The outgoing and incoming series practically coincide.

4. *Specific Inductive Capacity.*—In equation (7) if the space d' is filled with air, $K' = 1$. On the other hand if a plate of some insulator like glass is inserted of thickness d'_g

$$d' = d'_g + d'_a$$

where d'_a is the thickness of the air layer. Moreover if K_g is the specific inductive capacity of the insulator

$$d'/K' = d'_a + d'_g/K_g$$

If, therefore, in the absence of the insulator, y is the downward displacement of the upper plate which gives the same fringe displacement n, and hence the same V as the insertion of the insulator plate, the resulting equations eventually reduce to

$$K_g = d'_g/(d_g - y)$$

To determine the specific inductive capacity of a given insulating plate, the electrophorus is discharged at a convenient distance, d', between plate and hard rubber face. The insulator (K_g) is then inserted (noting the fringe displacement n) and withdrawn. The fringes must return to zero, showing that no charge has been imparted by the friction of the insulator. The upper plate is now depressed (y) on the micrometer screw until the same fringe displacement n is obtained. The operation is quite rapid; nevertheless the results so obtained were usually too large. Dielectric hysteresis was looked for, but could not have exceeded a fringe breadth.

5. *Absolute Values.*—The comparison of the U-tube with three different Elster and Geitel Electroscopes, the latter all standardized in volts, is given in figure 3 and is as linear within the reading error. The U-tube results were computed by equation A, measuring d from the mercury surface M' in figure 1 to the electrode C', with allowance for the K of glass plate. They are about four times too large. When, however, the measurement of d was made from the top of the glass plate to the electrodes, the results of the two instruments practically coincided. Hence the thin glass plate here acts like a conductor. The charge is transferred to its top face.

GEOMETRIC ASPECTS OF THE ABELIAN MODULAR FUNCTIONS OF GENUS FOUR (I)

BY ARTHUR B. COBLE[1]

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS

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1. *Introduction.*—The plane curve of genus 4 has a canonical series g_3^6 and is mapped from the plane by the canonical adjoints into the *normal* curve of genus 4, a space sextic which is the complete intersection of a quadric and a cubic surface. If we denote a point of this quadric by the parameters t, τ of the cross generators through it the equation of this sextic is $F = (a\tau)^3 (\alpha t)^3 = 0$. For geometric purposes we may define a modular function to be any rational or irrational invariant of the form F, bi-cubic in the digredient binary variables τ, t; for transcendental purposes it is desirable to restrict this definition by requiring further that this invariant, regarded as a function of the normalized periods ω_{ij} of the abelian integrals attached to the curve, be uniform.

There seems to be an unusually rich variety of geometric entities which center about this normal curve. Some of these have received independent investigation. It is the purpose of this series of abstracts to indicate a number of new relations among these various entities and to connect each with the normal sextic F. The methods employed are in the main geometric. Direct algebraic attack on problems which contain nine irremovable constants, or moduli, is difficult. However much information is gained by a free use of algebraic forms containing sets of variables drawn from different domains. Both finite and infinite discontinuous groups are utilized at various times.

2. *The Figure of Two Space Cubic Curves.*—White[2] has introduced for other purposes the interpretation of the form $F = 0$ as the incidence condition of the point τ of the space cubic curve $C_1(\tau)$ and the plane t of the space cubic $C_2(t)$. There is dually an incidence condition of plane τ of $C_1(\tau)$ and point t of $C_2(t)$, expressed by a form $\overline{F} = (A\tau)^3 (\mathrm{A}t)^3 = 0$. We call the sextics of genus 4 determined by $F = 0$ and $\overline{F} = 0$ *reciprocal.* Each is the same covariant of degree three of the other.

3. *A Set of Four Mutually Related Rational Plane Sextics.*—On each of the cubic curves $C_1(\tau)$, $C_2(t)$ regarded as a point locus there is a net of point quadrics Q_1, Q_2, respectively; on each regarded as a plane locus there is a net of quadric envelopes, $\overline{Q}_1$, $\overline{Q}_2$, respectively. The net Q_1 cuts the curve $C_2(t)$ in ∞^2 sets of six points which lie in an $I_2^6(t)$. An I_2^6 on a binary domain may be visualized as the line sections of a projectively definite (but not localized) rational plane sextic $S_2(t)$. Thus the four nets determine the four rational plane sextics of the array

$$\begin{array}{c} \underline{S}_1(\tau),\ \underline{S}_2(t) \\ \overline{S}_1(\tau),\ \overline{S}_2(t). \end{array}$$

Two sextics in a row of the array will be called *paired sextics;* two in a column, *counter sextics;* and the other pairs, *diagonal sextics.* If any one of these sextics be given, its I_2^6 spread out on a space cubic will determine the other space cubic and thereby the entire set of four. The nodal parameters of the paired sextics in the upper row are those of the ten common chords of C_1, C_2; in the lower row, those of the ten common axes of C_1, C_2. The equations of the sextics are

$$(\alpha\,\alpha')^2\,(\alpha t)\,(\alpha' t)\,(a\,\tau)^3\,(a'\tau)^3 = 0, \quad (AA')^2\,(A\tau)\,(A'\tau)\,(\mathrm{A}t)^3\,(\mathrm{A}'t)^3 = 0,$$

$$(A\,A')^2\,(\mathrm{A}\,t)\,(\mathrm{A}'t)\,(A\,\tau)^3\,(A'\tau)^3 = 0,\ (a\,a')^2\,(a\,\tau)\,(a'\tau)\,(\alpha\,t)^3\,(\alpha' t)^3 = 0.$$

Here the coefficients of the quadratics in t or τ furnish three line sections of the respective sextic. The significance of the quadratic parameter appears in 6.

4. *Two Birationally Related Quartic Surfaces.*—The two nets Q_1, Q_2 of point quadrics on C_1, C_2, respectively, are apolar to a web of quadric envelopes $\overline{Q}$; similarly the nets $\overline{Q}_1$, $\overline{Q}_2$ are apolar to a web of point quadrics, Q. The jacobians, $\overline{J}$, J, of these respective webs are quartic envelope or surface, respectively; the first on the ten common chords, the second on the ten common axes of C_1, C_2. If we map by means of the web Q its space upon another space, the jacobian J, the locus of nodes of quadrics in Q, is mapped upon a surface Σ of order 16 and class 4, the Cayley symmetroid quartic envelope with ten tropes. The two cubic curves are mapped upon two paired rational space sextics $\overline{R}_1(\tau)$, $\overline{R}_2(t)$ which are conjugate to the paired rational plane sextics $\overline{S}_1(\tau)$, $\overline{S}_2(t)$, respectively, i.e., plane sections of the space sextic are apolar to line sections of the *Conjugate* plane sextic. The symmetroid Σ is the locus of planes which cut the sextic $\overline{R}_i$ in catalectic sections. Similarly the jacobian $\overline{J}$ *counter* to J is mapped by the web $\overline{Q}$ upon a point symmetroid $\overline{\Sigma}$ *counter* to Σ, and C_1, C_2 upon rational space sextics $R_1(\tau)$, $R_2(t)$, counter to $\overline{R}_1(\tau)$, $\overline{R}_2(t)$, respectively, and conjugate to $S_1(\tau)$, $S_2(t)$, respectively.

5. *References.*—Meyer[3] has discussed the relation of J to the sextic $\overline{S}_2(t)$ and mentions the occurrence of counter sextics. Conner[4] considers the mapping from J to Σ and its connection with the paired rational sextics. The above introduction of the tetrad of rational sextics as defined by the sextics F, $\overline{F}$ of genus 4 is novel. Schottky,[5] beginning with the abelian theta functions of genus 4, derives a set of ten points in space which are the nodes of a quartic surface and merely states a characteristic property of this surface by which it can be identified with Σ. The writer[6] has shown that Σ can be transformed by regular Cremona transformation into only a finite number of projectively distinct symmetroids. These classes permute under the group (mod. 2) of integer transformations of the periods of the functions of genus 4. The analogous result for the plane

rational sextic involves a subgroup of the group (mod. 2) of the periods of the functions of genus 5. This indicates a connection (which we seek) of the functions of genus 4 and those of genus 5. Proceeding the other way Wirtinger[7] obtains the plane sextic of genus 4 as the locus of vertices of diagonal triangles of a linear series g_1^4 upon a ternary quartic ($p = 3$). This transition will be discussed later.

6. *The Covariant Conic* $\overline{R}(\tau)$ *of the Rational Plane Sextic* $\overline{S}_2(t)$.—From the existence in the net $\overline{Q}_1$ of a quadratic system of cones we conclude that the rational sextic $\overline{S}_2(t)$ has a covariant conic $K(\tau)$ such that the ten nodes of $S_2(t)$ determine upon $K(\tau)$ the ten pairs of nodal parameters of the sextic $\overline{S}_1(t)$ paired with the given sextic $\overline{S}_2(t)$. This theorem furnishes the bond between ten nodes as a ternary figure and ten nodes as a binary figure on the rational curve. The equation of the sextic in Darboux coördinates referred to the norm conic $K(\tau)$ is precisely that given in 3.

7. *The Perspective Cubics of* $\overline{S}_2(t)$. *The form* $\binom{113}{211}$. We denote by the symbol $\binom{i_1,\, i_2, \cdots\cdots}{k_1,\, k_2, \cdots\cdots}$ an algebraic form of order i_1 in the variables of an S_{k1}, of order i_2 in the variables of an S_{k2}, etc. Unless explicitly restricted these sets of variables are digredient. Thus $F = (a\tau)^3\ (\alpha t)^3$ is a form $\binom{33}{11}$. By polarizing F into $(a\tau_1)\ (a\tau_2)\ (a\tau)\ (\alpha t)^3$ and replacing the pair of parameters τ_1, τ_2 by the point x which they determine in the plane of $K(\tau)$ we obtain the $\binom{113}{211}$ form

$$(\pi x)\ (d\tau)\ (\delta t)^3$$

a general form of the orders indicated with nine absolute constants. For given τ this form determines a rational cubic envelope *perspective*[8] to the sextic $\overline{S}_2(t)$, i.e., line t of the cubic is on point t of the sextic. The sextic is the locus of the meets of corresponding lines of any two of the ∞^1 perspective cubics, and it has the equation $(\pi\pi'\xi)\ (dd')\ (\delta t)^3\ (\delta' t)^3 = 0$. Conversely given the sextic the family of perspective cubics is determined. Each cubic τ has three cusps whose parameters are given by $(\pi\pi'\pi'')\ (d\tau)\ (d'\tau)\ (d''\tau)\ (\delta\delta')^3\ (\delta'' t)^3 = 0$. This is $\overline{F} = (A\tau)^3\ (\mathrm{A}t)^3$ whence the cusp locus, $\overline{G}C(\tau)$, is birationally general and of genus 4. The equation of the cusp locus is the determinant of the coefficients of $(\pi x)\ (\pi' x)\ (d\tau)\ (d'\tau)\ (\delta\delta')^2\ (\delta t)\ (\delta' t)$ regarded as a form bi-quadratic in τ, t. Thus $\overline{G}C(\tau)$ is a sextic whose six nodes are the points for which the first minors of the above determinant vanish and these first minors furnish the nine linearly independent quartic adjoints of $\overline{G}C(\tau)$.

The curve of genus 4 has two special series g_1^3, residual with respect to each other in the canonical series. These appear in the normal form as the triads on the two sets of generators of the quadric containing the sextic. One of these series on $\overline{G}C(\tau)$ is the triads of cusps of perspective cubics of $\overline{S}_2(t)$. The web of adjoint cubics of $\overline{G}C(\tau)$ is furnished by the form

$$(\pi x)\ (\pi' x)\ (\pi'' x)\ (dd')\ (\delta\delta')\ (\delta'\delta'')\ (\delta\delta'')^2\ (\delta' t)\ (d''\tau) = 0,$$

t and τ being variable with the cubic of the web. For fixed τ and variable

t we have the pencil of adjoint cubics on the cusp triad of the perspective cubic τ.

The form $(\pi x)\ (d\tau)\ (\delta t)^3$ for fixed x and variable τ is a pencil of binary cubic. This pencil has two linear combinants[9]: $\bar{a} = (\pi x)\ (\pi' x)\ (dd^1)\ (\delta\delta')\ (\delta t)^2\ (\delta' t)^2$ and $\bar{b} = (\pi x)\ (\pi' x)\ (dd')\ (\delta\delta')^3$. The invariants i, j of the binary quartic $\bar{a}$ also are combinants. The invariant j is a sextic curve $\bar{d}$, the invariant i is $\bar{b}^2$. Hence the discriminant of $\bar{a}$ factors and the two factors $\bar{b}^3 + \bar{d}$ and $\bar{b}^3 - \bar{d}$ furnish the equations of the cusp locus $\bar{G}C(\tau)$ and the rational sextic $\bar{S}_2(t)$. We conclude further that there are 12 perspective cubics of $S_2(t)$ with flex points at the meets of $\bar{b}$ and $\bar{d}$. The sextics osculate at these points with the flex tangents as common tangents. The 12 flex points are the branch points on $\bar{G}C(\tau)$ of the function $t(\tau)$ defined by $\bar{F} = 0$. Thus a projective (but not a birational) peculiarity of $\bar{G}C(\tau)$ is that the 12 branch points of one of its series g_1^3 lie on a conic $\bar{b}$.

The parametric line equation of the conic $K(\tau)$ on which the nodes of $\bar{S}_2(t)$ determine the nodal parameters of $\bar{S}_1(\tau)$ is of degree four in the coefficients of $\binom{113}{211}$. Its symbolic form is $(\pi\pi'\pi'')\ (\pi''' x)\ (d'\tau)\ \ (\delta\delta')^3\ \ (\delta''\delta''')^3$ $\{\ (d\tau)\ (d''d''') + 2(d''\tau)\ (dd''')\ \} = 0$.

With reference to the cubic space curves $C_1(\tau)$, $C_2(t)$ the point x determines an axis l_x of $C_1(\tau)$ on planes τ_1, τ_2; τ is the third plane of $C_1(\tau)$ on a point y of l_x; and t a plane of $C_2(t)$ on y. Then to points x on $\bar{G}C(\tau)$ there correspond axes of C_1 on points of C_2 and to the nodes of $\bar{G}C(\tau)$ the six axes of C_1 which are chords of C_2; to points x on $\bar{S}_2(t)$ there correspond axes of C_1 on planes of C_2, and to the nodes of $\bar{S}_2(t)$ the ten common axes of C_1, C_2. If $x_\circ$ is a node of $\bar{S}_2(t)$ the form $(\pi x_\circ)\ (d\tau)\ (\delta t)^3$ factors into $(l_\circ \tau)\ (\lambda_\circ t)\ .\ (q_\circ t)^2$ where $(q_\circ t)^2$ is the pair of nodal parameters. The ten forms $(l_\circ \tau)\ (\lambda_\circ t)$ will appear later in connection with the symmetroid. Other covariants of the $\binom{113}{211}$ form are easily interpretable with reference to C_1, C_2. Thus $\bar{a}$ furnishes the four parameters t of tangents of C_2 which meet the axis l_x of C_1, and b determines the axes l_x of C_1 which are in the null system of C_2.

From the definition of perspective curves the line t' of the perspective cubic $(\pi x)\ (d\tau)\ (\delta t')^3$ will cut the sextic $(\pi\pi'\xi)\ (\delta t)^3\ (\delta' t)^3\ (dd')$ in the point $t = t'$. On forming the incidence condition of line and point, removing the factor (tt'), and setting $t' = t$, we obtain

$$(\pi\pi'\pi'')\ (\delta\delta')\ (\delta t)^2\ (\delta' t)^2\ (\delta'' t)^3\ (d'd'')\ (d\tau)$$

which furnishes the seven contacts[10] t of the perspective cubic with the sextic. This is a form $\binom{71}{11}$ of general type containing nine absolute constants which will appear later.

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[1] This investigation has been pursued under the auspices of the Carnegie Institution of Washington, D. C.

[2] H. S. White, these PROCEEDINGS, **2**, 1916 (337).

[3] Meyer, *Apolarität und Rationale Curven*, pp. 320–47.

[4] Conner, *Amer. J. Math.*, **37**, 1915 (29).

[5] Schottky, *Acta Math.*, **27**, 1903 (235).

[6] Coble, *Amer. J. Math.*, **41**, 1919 (243).

[7] Wirtinger, *Math. Ann.*, **27**, 1892 (261); *Untersuchung über Thetafunctioners*, Leipzig (1895).

[8] Coble, *Amer. J. Math.*, **32**, 1910 (333).

[9] Shenton, *Ibid.*, **372** 1915 (247).

[10] Cf. Coble, *l. c.*, p. 352.

MELANOVANADITE, A NEW MINERAL FROM MINA RAGRA, PASCO, PERU

By Waldemar Lindgren

Department of Mining, Metallurgy and Geology, Massachusetts Institute of Technology

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Late in 1920 Mr. W. Spencer Hutchinson, Consulting Engineer for the Vanadium Company of America, brought to my attention three specimens of a mineral collected by him at Mina Ragra, Peru. He suspected that it was a new mineral, and this opinion was proved correct by chemical and optical examination. The formula is $2CaO.3V_2O_5.2V_2O_4$ and I wish to propose for it the name of Melanovanadite, in allusion to it being practically the only vanadium mineral of a deep black color.[1]

The mineral occurs in acicular bunches on black brecciated shale, the individual crystals being at most 3 mm. long.

The greater thickness of the needles is about 0.5 mm. ranging down to 0.1 and 0.01 mm. The color is black, luster almost submetallic, streak very dark reddish brown. The hardness is 2.5 the specific gravity 3.477 at 15° C. The habit of the crystals is prismatic, parallel to *c*, with monoclinic symmetry. The principal faces consist of a flat, striated prism, the longer diagonal being parallel to the *b* axis, minor pinacoidal faces, and usually well developed terminal faces of pyramids and smaller domes. The crystals have a perfect cleavage parallel to (010).

Under the microscope the crystals remain black except in very thin prisms which are translucent with brown color.

Flat cleavage pieces parallel to the clinopinacoid only become translucent when the thickness is about 0.003 mm. and then show maximum extinction of about 15 °. Resting on the prism (100) the crystals become brown translucent with a thickness of about 0.03 mm. and then show lower extinctions of 12° to 13°, while these resting more nearly on the orthopinacoid extinguish at lower angles. The perfect cleavage being perpendicular to the poorly developed orthopinacoid, extinctions of 0° are rarely seen. Exact optical measurements are difficult on account of the deep color. Obscure hyperbolae show on the prism faces and it is probable that the plane of the optic axes lies parallel to the perfect cleavage (010). The absorption is very strong. The α ray is visible with dark yellowish brown color through the prism faces and the orthopinacoid and has according to a determination kindly made by Professor C. H. Warren a coefficient of refraction of 1.74; while the β and γ rays are somewhat higher but cannot be measured exactly on account of the strong absorption. The β and γ rays therefore lie in the 010 plane of perfect cleavage and their absorption is so strong that such cleavage pieces only become translucent in extremely thin plates, with dark reddish brown color, β and γ differing slightly in depth of tint. The double refraction is strong.

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[1] The ending "vanadite" is an obsolete form of "vanadinite," but there can scarcely be any objection to using this form in the present case.

The material was analysed by Professor L. F. Hamilton of the Massachusetts Institute of Technology, who reported the following composition:

V_2O_5	52.61
V_2O_4	33.34
CaO	9.89
MgO	.27
$Al_2O_3Fe_2O_3$	1.89
Silica	1.66
	99.66

Assuming the alumina, iron oxide, magnesium and silica to be impurities from the admixed shale the analysis may be recalculated to:

V_2O_5	54.90
V_2O_4	34.78
CaO	10.32
	100.00

From this we calculate

$$V_2O_5 \quad \frac{54.90}{182} = 0.302$$

$$V_2O_4 \quad \frac{34.78}{166} = 0.209$$

$$CaO \quad \frac{10.32}{56} = 0.184$$

$$V_2O_5 : V_2O_4 : CaO = 302 : 209 : 184$$

or approximately 3:2:2. The formula would therefore be

$$2\,CaO.3V_2O_5.2V_2O_4$$

The calculated composition of this would be:

V_2O_5	55.16
V_2O_4	33.54
CaO	11.30
	100.00

Before the blowpipe the mineral fuses very easily to a brown liquid but gives no flame reaction for calcium. It gives strong bead reactions for vanadium. It is easily soluble, with apple green color, in HNO_3, HCl, and dilute H_2SO_4. Upon the evaporation the solution gives a red or brown residue. In potassium hydroxide melanovanadite is easily soluble with brownish color.

Editorial Note:—This issue of the PROCEEDINGS, those immediately preceding, and those immediately to follow, have been seriously delayed by the nation-wide printers' strike which began on May first, 1921.

The Editors and the publisher of the PROCEEDINGS desire to express their gratitude to the authors and to the subscribers for the patience which has been shown in respect to these trying delays and to assure them that all efforts are being made by all concerned toward getting the PROCEEDINGS back to regular and prompt publication with every indication that success will be assured in the very near future.

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THE AVERAGE OF AN ANALYTIC FUNCTIONAL[1]

By Norbert Wiener
Department of Mathematics, Massachusetts Institute of Technology

Communicated by A. G. Webster, March 22, 1921

1. Conceive a particle free to wander along the x-axis. Suppose the probability that it wander a given distance independent

(1) of the position from which it starts to wander,
(2) of the time when it starts to wander,
(3) of the direction in which it wanders.

It may be shown[2] that under these circumstances, the probability that after a time, t, it has wandered from the origin to a position lying between $x = x_0$ and $x = x_1$ is

$$\frac{1}{\sqrt{\pi c t}}\int_{x_0}^{x_1} e^{-\frac{x^2}{ct}}\,dx$$

where t is the time and c is a certain constant which we can reduce to 1 by a proper choice of units. This choice we shall make in what follows. The exponential form of this integral needs no comment, while the mode in which t enters results from the fact that

$$\frac{1}{\sqrt{\pi(t_1+t_2)}}\int_{x_0}^{x_1} e^{-\frac{x^2}{t_1+t_2}}\,dx = \frac{1}{\sqrt{\pi t_1}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{\pi t_2}}\int_{x_0}^{x_1} e^{-\frac{(y-x)^2}{t_2}}\,dy\right]e^{-\frac{x^2}{t_1}}\,dx$$

This identity will be presupposed in all that follows.

Let us now consider a particle wandering from the origin for a given period of time, say from $t=0$ to $t=1$. Its position will then be a function of the time, say $x = f(t)$. There are certain quantities—functionals—which may depend on the whole set of values of f from $t=0$ to $t=1$. If we take a large number of particles (i.e. a large number of values of f) at random, it is natural to suppose that the average value of the functional will often approach a limit, which we may call the average value of the functional over its entire range. What will this average be, and how shall we find it?

If $F\{f\}$ is a functional depending on the values of f for only a finite number of values of the argument of f—if $F\{f\}$ is a function[3] of $f(t_1)$, $f(t_2)$,…, $f(t_n)$ of the form $\Phi[f(t_1), \ldots, f(t_n)]$—then it is easy enough to give a natural definition of the average of F, which we shall write $A\{F\}$. We can reasonably say

$$A\{F\} = \frac{1}{\sqrt{\pi^n t_1(t_2-t_1)\ldots(t_n-t_{n-1})}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi(x_1, \ldots, x_n) \\ e^{-\frac{x_1^2}{t_1} - \frac{(x_2-x_1)^2}{t_2-t_1} - \ldots - \frac{(x_n-x_{n-1})^2}{t_n-t_{n-1}}} dx_1 \ldots dx_n.$$

In particular if $\quad F\{f\} = [f(t_1)]^{m_1}[f(t_2)]^{m_2} \ldots [f(t_n)]^{m_n},$
then

$$A\{F\} = \frac{1}{\sqrt{\pi^n t_1(t_2-t_1) \ldots (t_n-t_{n-1})}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} x_1^{m_1} \ldots x_n^{m_n} \\ e^{-\frac{x_1^2}{t_1} - \ldots - \frac{(x_n-x_{n-1})^2}{t_n-t_{n-1}}} dx_1 \ldots dx_n$$

$$= \frac{1}{\sqrt{\pi^n t_1(t_2-t_1) \ldots (t_n-t_{n-1})}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} y_1^{m_1}(y_2+y_1)^{m_2} \ldots (y_n + y_{n-1} + \ldots \\ + y_1)^{m_n} e^{-\frac{y_1^2}{t_1} - \ldots - \frac{y^2_n}{t_n-t_{n-1}}} dy_1 \ldots dy_n$$

$$= \frac{1}{\sqrt{\pi^n}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (t_1^{1/2} z_1)^{m_1} \left[(t_2-t_1)^{1/2} z_2 + t_1^{1/2} z_1\right]^{m_2} \ldots \left[(t_n-t_{n-1})^{1/2} z_n \right. \\ \left. + \ldots + t_1^{1/2} z_1\right]^{m_n} e^{-z_1^2 - \ldots - z_n^2} dz_1 \ldots dz_n.$$

This latter integral is in the form

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} P(z_1, \ldots, z_n) e^{-z_1^2 - \cdots - z_n^2} dz_1, \ldots dz_n$$

where P is a polynomial, and can be evaluated by means of the well known formulae:

$$\int_{-\infty}^{\infty} e^{-y^2} y^{2n+1} dy = 0,$$

$$\int_{-\infty}^{\infty} e^{-y^2} y^{2n} dy = \sqrt{\pi} \, \frac{1.3.5\ldots(2n-1)}{2^n}.$$

We can thus easily evaluate $A\{F\}$ as a polynomial in $t_1, t_2, \ldots, t_n$, which we shall call $P_{m_1, \ldots, m_n}(t_1, \ldots, t_n)$. It is easy to show that if Σm_k is odd,

$$P_{m_1, \ldots, m_n}(t_1, \ldots, t_n) = 0.$$

To return to the more general functional: there is a large class of so-called analytic functionals,[4] which may be expanded in the form of series such as

$$F\{f\} = a_0 + \int_0^1 f(x)\varphi_1(x)dx + \int_0^1 \int_0^1 f(x)\, f(y)\varphi_2(x,y)dxdy + \dots$$
$$+ \int_0^1 \dots \int_0^1 f(x_1) \dots f(x_n)\varphi_n(x_1, \dots, x_n)\, dx_1 \dots dx_n + \dots$$

and an even wider class of what may be called Stieltjes analytic functionals, in which the general term

$$\int_0^1 \dots \int_0^1 f(x_1) \dots f(x_n)\varphi_n(x_1, \dots, x_n)dx_1, \dots dx_n$$

is replaced by the Stieltjes integral[5]

$$\int_0^1 \dots \int_0^1 f(x_1) \dots f(x_n)d\psi_n(x_1, \dots, x_n).$$

In what follows, we shall confine our discussion to Stieltjes analytic functionals, which we shall call simply analytic. The problem with which we are now concerned is the definition of the average of an analytic functional. Now, the first property which any average ought to fulfil is that the average of the sum of two functionals should equal the sum of their averages. We should expect, therefore, that:

(*a*) Over a wide range of cases, the average of a series should equal the series of the averages of the terms;

(*b*) The average of a Stieltjes integral, single or multiple, of a given functional with respect to such parameters as it may contain, should be equal to the integral of the average;

(*c*) A constant multiplied by the average of a functional should equal the average of a constant times the functional.

In accordance with this, we get the following natural definition of the average of the analytic functional F.

$$A\{F\} = A\{a_0 + \int_0^1 f(x)d\psi_1(x) + \int_0^1 \int_0^1 f(x)f(y)d\psi_2(x,y) + \dots\}$$
$$= a_0 + A\left\{\int_0^1 f(x)d\psi_1(x)\right\} + A\left\{\int_0^1 \int_0^1 f(x)\, f(y)d\psi_2(x,y)\right\} + \dots$$
$$= a_0 + \int_0^1 A\{f(x)\}d\psi_1(x) + \int_0^1 \int_0^1 A\{f(x)f(y)\}d\psi_2(x,y) + \dots$$

We have already seen how to determine $A\{f(x_1)\dots f(x_n)\}$ as a polynomial in the x_k's. Hence whenever the above series converges, we have now a way of obtaining a perfectly definite value for $A\{F\}$. It may be noted that every term in the above expression in which the sign of integration is repeated an odd number of times is identically zero.

If $A\ \{F\}$ is to behave as we should expect it to behave, there are certain properties which it must fulfil, at least over a large and important class of cases. Among these are the following:

$$(1) \qquad A\{F_1\} + A\{F_2\} = A\{F_1 + F_2\}$$

$$(2) \qquad cA\{F_1\} = A\{cF_1\}$$

$$(3) \qquad \sum_1^\infty A\{F_n\} = A\left\{\sum_1^\infty F_n\right\}$$

(4) If F_x is a functional depending on the parameter x, and $u(x)$ is a function of limited total variation, then

$$\int_a^b A\{F_x\}du = A\left\{\int_a^b F_x du\right\}$$

(5) Suppose $F_{t_1, \ldots, t_n}(x_1, \ldots, x_n)$ be defined as $F\left\{f_{t_1, \ldots, t_n}^{x_1, \ldots, x_n}(t)\right\}$, where $f_{t_1, \ldots, t_n}^{x_1, \ldots, x_n}(t)$ assumes the value

$$x_k + \frac{(t - t_k)\,(x_{k+1} - x_k)}{t_{k+1} - t_k}$$

for $t_k \leq t \leq t_{k-1}$. Then

$$A\{F\} = lim\, \pi^{-\frac{n}{2}} \prod_1^n (t_k - t_{k-1})^{-1/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} F_{t_1, \ldots, t_n}(x_1, \ldots, x_n)$$
$$e^{-\frac{x_1^2}{t_1} - \sum_2^n \frac{(x_k - x_{k-1})^2}{(t_k - t_{k-1})}} dx_1 \ldots dx_n,$$

where the limit is taken as the t_k's increase in number in such a manner to divide the interval from 0 to 1 more and more finely.

2. The next task before us is to investigate hypotheses which are sufficient to guarantee the validity of propositions (1)–(5). Propositions (1) and (2) require indeed very little discussion, for they are always satisfied when the series for F_1, F_2, $A\{F_1\}$ and $A\{F_2\}$ converge. In (3), let $F_1, \ldots, F_n, \ldots$, and the series ΣF_n all possess averages, and let

$$\sum_1^\infty F_n = \sum_1^m F_n + R_m$$

where $A\{R_m\}$ vanishes as m increases without limit. Then

$$A\left\{\sum_1^\infty F_n\right\} = \sum_1^m A\{F_n\} + A\{R_m\}.$$

Therefore

$$\lim_{m=\infty} \left| A\left\{\sum_1^\infty F_n\right\} - \sum_1^m A\left\{F_n\right\}\right| = 0$$

and (3) is proved. If ΣF_n converges and $lim\, A\{R_m\} = 0$, we shall say ΣF_n converges *smoothly*.

Proposition (4) reduces to the ordinary inversion of a multiple Stieltjes integral when $F_x\{f\}$ is of the form

$$\int_0^1 \cdots \int_0^1 f(x_1) \ldots f(x_n) d\psi_n(x_1, \ldots, x_n, x)$$

and ψ_n is a function of limited variation in $x_1, \ldots, x_n$. What we wish to prove is that

$$\int_a^b \left\{ \int_0^1 \cdots \int_0^1 A \{ f(x_1) \ldots f(x_n) \} d\psi_n(x_1, \ldots, x_n) \right\} du$$

$$= A \left\{ \int_a^b \left\{ \int_0^1 \cdots \int_0^1 f(x_1) \ldots f(x_n)\, d\psi_n\, (x_1, \ldots, x_n) \right\} du \right\}$$

Now

$$\int_a^b \left\{ \int_0^1 \cdots \int_0^1 f(x_1) \ldots f(x_n)\, d\psi_n\, (x_1, \ldots, x_n) \right\} du =$$

$$\int_a^b \left[lim \sum_{\alpha, \ldots, \vartheta} f(\xi_{1\alpha}) \ldots f\, (\xi_{n\vartheta})\, \Delta_{x_{1\alpha} \ldots x_{n\vartheta}}^{x_{1,\, \alpha+1} \ldots x_{n,\, \vartheta+1}} \psi_n(x_1, \ldots, x_n, x) \right] du.$$

If in this latter expression the total variation of ψ_n is less than a quantity independent of x, we can permute the $\int_a^b$ and the *lim*, and get

$$lim \sum_{\alpha, \ldots, \vartheta} f(\xi_{1\alpha}) \ldots f(\xi_{n\vartheta})\, \Delta_{x_{1\alpha} \ldots x_{n\vartheta}}^{x_{1,\, \alpha+1} \ldots x_{n,\, \vartheta+1}} \int_a^b \psi_n\, (x_1, \ldots, x_n, x)\, du,$$

which we may write

$$\int_0^1 \cdots \int_0^1 f(x_1) \ldots f(x_n) d \left[\int_a^b \psi_n(x_1, \ldots, x_n, x)\, du \right].$$

In this we suppose f uniformly continuous. It is easy to show that on our assumptions $\int_a^b \psi_n(x_1, \ldots, x_n, x) du$ is of limited variation. Consequently we obtain

$$A \left\{ \int_a^b \left\{ \int_0^1 \cdots \int_0^1 f(x_1) \ldots f(x_n) d\psi_n(x_1, \ldots, x_n, x) \right\} du \right\}$$

$$= A \left\{ \int_0^1 \cdots \int_0^1 f(x_1) \ldots f(x_n) d \left[\int_a^b \psi_n(x_1, \ldots, x_n, x) du \right] \right\}$$

$$= \int_0^1 \cdots \int_0^1 A \{ f(x_1) \ldots f(x_n) \} d \left[\int_a^b \psi_n(x_1, \ldots, x_n, x)\, du \right].$$

A further transformation just like the preceding turns this into

$$\int_a^b \left\{ \int_0^1 \cdots \int_0^1 A \{ f(x_1) \ldots f(x_n) \} d\psi_n\, (x_1, \ldots, x_n, x) \right\} du$$

so that we have now a sufficient condition for the validity of our theorem. The extension to non-homogeneous terminating analytic functionals is obvious. The extension to non-terminating analytic functionals may be deduced with the help of (3) and a well-known theorem on the integration of uniformly convergent series, and reads as follows: *let* F_x, *be an analytic functional of the form*

$$a_0 + \sum_1^\infty \int_0^1 \cdots \int_0^1 f(x_1) \ldots f(x_n)\, d\psi_n(x_1, \ldots, x_n, x)$$

where the total variation of each ψ_n *is less than some quantity independent of* x, *and let each* ψ_n *be uniformly continuous in* x *over the interval* (a, b). *Let* $u(x)$ *be a function of limited total variation in* x *over the same interval. Let* $A\left\{\int_a^b F_x du\right\}$ *exist, and let*

$$\lim_{m=\infty} \sum_{n=m}^{\infty} \int_0^1 \ldots \int_0^1 A\{f(x_1) \ldots f(x_n)\}\, d\psi_n\,(x_1, \ldots, x_n, x) = 0$$

uniformly in x. *Then*

$$\int_a^b A\{F_x\} du = A\left\{\int_a^b F_x\, du\right\}.$$

As to (5), let us begin as above with a functional of the form

$$\Phi\{f\} = \int_0^1 \ldots \int_0^1 f(x_1) \ldots f(x_n)\, d\psi_n\,(x_1, \ldots, x_n).$$

Consider

$$\mathrm{I} = \int_0^1 \ldots \int_0^1 A\{f(x_1) \ldots f(x_n\}\, d\psi_n\,(x_1, \ldots, x_n),$$

where A is taken in the original sense as an n-fold integral. By definition

$$\mathrm{I} = \lim \sum_{\alpha, \ldots, \vartheta} A\{f(\xi_{1\alpha}) \ldots f(\xi_{n\theta})\}\, \Delta_{x_{1\alpha} \ldots x_{n\vartheta}}^{x_{1,\alpha+1} \ldots x_{n,\vartheta+1}} \psi_n(x_1, \ldots, x_n),$$

where $x_{ko} = 0, x_{k1}, \ldots, x_{k\mu} = 1$ is an increasing sequence of numbers, ξ_{kK} lies between x_{kK} and $x_{k,K+1}$ and *lim* is taken as $max(x_{k,K+1} - x_{kK})$ approaches 0. Let V be the total variation of ψ_n as its arguments range from 0 to 1, and let M stand for $max(x_{k,K+1} - x_{kK})$. Let Q stand for the least upper bound of the variation of $A\{f(x_1) \ldots f(x_n)\}$ as the point $(x_1, \ldots, x_n)$ wanders over an interval $\left(\begin{matrix} x_{1,\alpha+1} \ldots, x_{n\vartheta+1} \\ x_{1\alpha} \ldots x_{n\vartheta} \end{matrix}\right)$. Then

$$(6) \quad \left|\mathrm{I} - A\left\{\sum_{\alpha, \ldots, \vartheta} f(\xi_{1\alpha}) \ldots f(\xi_{n\vartheta}) \Delta_{x_{1\alpha} \ldots x_{n\vartheta}}^{x_{1,\alpha+1} \ldots x_{n,\vartheta+1}} \psi_n(x_1, \ldots, x_n)\right\}\right| \leq VQ.$$

Now, let $f_m(x)$ be that function whose graph is the broken line with corners at $(x_{(K)}, f(x_{(K)}))$, where $0 \leq K \leq \mu, x_{(o)} = 0, x_{(\mu)} = 1$. Then if x lies between $x_{(K)}$ and $x_{(K+1)}, f_m(x)$ is of the form $af_m(x_{(K)}) + bf_m(x_{(K)}) \div a + b$. It follows that if $(\xi_{1\alpha}, \ldots, \xi_{n\vartheta})$ lies in the interval $\left(\begin{matrix} x_{(\alpha+1)} \ldots x_{(\vartheta+1)} \\ x_{(\alpha)} \ldots x_{(\vartheta)} \end{matrix}\right)$, $A\ \{f_m(\xi_{1\alpha}) \ldots f_m(\xi_{n\vartheta})\}$ is of the form

$$\frac{a_1 C_1 + a_2 C_2 + \ldots + a_\rho C_\rho}{a_1 + a_2 + \ldots + a_\rho}$$

where each C_k is the value of some $A\{f(\xi_{1\alpha}) \ldots f(\xi_{\eta\vartheta})\}$ such that $(\xi_{1\alpha}, \ldots, \xi_{\eta\vartheta})$ is a corner of the hyperparallelopiped $\left(\begin{matrix} x_{(\alpha+1)} \ldots x_{(\vartheta+1)} \\ x_{(\alpha)} \ldots x_{(\vartheta)} \end{matrix}\right)$. It readily results from considerations of continuity that $A\{f_m(\xi_1) \ldots f_m(\xi_\mu)\}$ is of the form $A\{f(\eta_{1\alpha}) \ldots f(\eta_{n\vartheta})\}$, where $(\eta_{1\alpha}, \ldots, \eta_{n\vartheta})$ also lies in the interval $\left(\begin{matrix} x_{(\alpha+1)} \ldots x_{(\vartheta+1)} \\ x_{(\alpha)} \ldots x_{(\vartheta)} \end{matrix}\right)$.

Making use of this fact, and of the fact that

$$\frac{1}{\sqrt{\pi^n t_1(t_2 - t_1)\ldots(t_n - t_{n-1}}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Phi(x_1, \ldots, x_n)\, e^{-\frac{x_1^2}{t_1} - \ldots - \frac{(x_n - x_n - 1)^2}{t_n - t_n - 1}}\, dx_1 \ldots dx_n$$

is an increasing functional of $\Phi(x_1, \ldots, x_n)$, we can draw the conclusion that

$$A\{\Phi\{f_m\}\} = A\left\{\int_0^1 \cdots \int_0^1 f_m(x_1) \ldots f_m(x_n)\, d\psi_n(x_1, \ldots, x_n)\right\}$$

if it exists, lies between the uppermost and lowermost values of

$$\sum_{\alpha, \ldots, \vartheta} A\{f_m(\xi_{1\alpha}) \ldots f_m(\xi_{n\vartheta})\}\, \Delta_{x(\alpha) \ldots x(\vartheta)}^{x(\alpha+1) \ldots x(\vartheta+1)} \psi_n(x_1, \ldots, x_n)$$

and hence of

$$\sum_{\alpha, \ldots, \vartheta} A\{f(\eta_{1\alpha}) \ldots f(\eta_{n\vartheta})\}\, \Delta_{x(\alpha) \ldots x(\vartheta)}^{x(\alpha+1) \ldots x(\vartheta+1)} \psi_n(x_1, \ldots, x_n)$$

From this and (6) we can deduce

$$(7) \qquad |\mathrm{I} - A\{\Phi\{f_m\}\}| \leq 2VQ$$

where Q is taken for $x_{kK} = x_{(K)}$.

This proves our theorem for homogeneous analytic functionals. In precise terms, then, our general theorem will read: *let*

$$F\{f\} = a_0 + \sum_1^{\infty} \int_0^1 \cdots \int_0^1 f(x_1) \ldots f(x_n)\, d\psi_n(x_1, \ldots, x_n) = a_0 + \sum_1^{\infty} F_n\{f\}$$

be an analytic functional. Let V_n stand for the total variation of F_n as its arguments range from 0 to 1: Let $(x_{(0)}, \ldots, x_{(\mu)})$ be a set of numbers in ascending order from 0 to 1, inclusive. Let M stand for max $(x_{(K+1)} - x_{(K)})$ and Q_n for the upper bound of the variation of $A\{f(x_1) \ldots f(x_n)\}$ as the point $(x_1, \ldots, x_n)$ wanders over the interval $\binom{x(\alpha+1) \ldots x(\vartheta+1)}{x(\alpha) \ldots x(\vartheta)}$. Let $f_\mu(x)$ be the function whose graph is the broken line with corners at (0, 0) and $(x_{(K)}, f(x_{(K)}))$. Then if

$$(a) \qquad \lim_{M=0} \sum_1^{\infty} V_k Q_k = 0,$$

(b) $A\{F_n\{f_\mu\}\}$ exists for every μ and n according to the definition of A as a multiple integral:

(c) the series for F converges smoothly;

(d) $\lim\limits_{n=\infty} A\left\{\sum_n^{\infty} F_m\{f_\mu\}\right\}$ exists for every μ, when A is taken as a multiple integral; it follows that

$$A\{F\} = \lim_{M=0} A\{F\{f_m\}\},$$

where the first A is defined in the sense of the average of an analytic functional, and the second as a multiple integral. A precisely analogous theorem holds when $f_m(x)$ instead of a broken straight line is any broken line with corners at $(0, 0)$ and at $(x_{(K)}, f(x_{(K)}))$, and consists of monotone arcs between these points. This last theorem makes our average of a functional the limit of the average of a function of a discrete set of variables, and justifies our use of the term average.

[1] The problem of the mean of a functional has been attached by Gâteaux (*Bull. Soc. Math. de France*, 1919, pp. 47–70). The idea of using the analytic functional as a basis is there found. The actual definition, however, is essentially different, and does not lend itself readily to the treatment of the Brownian Movement, for which the present method is especially adapted.

[2] Einstein, *Leipzig, Annalen Physik*, **17**, 905.

[3] We here take $t_1, < t_2 < \ldots < t_n$.

[4] Cf. V. Volterra, *Fonctions des Lignes.*

[5] Cf. P. J. Daniell, *Annals of Mathematics*, Sept., 1919, p. 30.

ON THE CALCULATION OF THE X-RAY ABSORPTION FREQUENCIES OF THE CHEMICAL ELEMENTS

By William Duane

Jefferson Physical Laboratory, Harvard University

Communicated July 23, 1921

The K critical absorption frequency of a chemical element is the highest frequency of vibration known to be characteristic of that element. In our laboratory we have measured the K critical absorption frequencies of most of the chemical elements by the ionization method. This data, together with measurements made elsewhere by the photographic method, may be found in Table 2 of a report by the author on Data Relating to X-Ray Spectra, which has been published by the National Research Council.

At a symposium on Ultra-Violet Light and X-Rays, held at the meeting of the American Association for the Advancement of Science at St. Louis in December 1919,[1] I presented a set of computations of the K critical absorption frequencies based on the Rutherford-Bohr theory of atomic structure and the mechanism of radiation. The computed values equalled the observed values to within one or two per cent. In these computations the electrons were supposed to revolve in orbits which lay in planes passing through the nucleus of the atom.

Later I presented[2] to the National Academy of Sciences and to the American Physical Society computations of these K critical absorption frequencies, calculated on the assumption that the orbits did not all lie in planes through the nucleus. I assumed that the orbits were circular, that they occurred in pairs and that the two orbits in a pair lay opposite to each other in parallel planes equi-distant from the nucleus, as represented in the figure. This gives a volume distribution of electrons. The mutual repulsion of the electrons for each other keep the orbits apart. It is necessary, however, to suppose that the electrons in the two orbits of a pair revolve in opposite directions. Otherwise they would be pulled together to form a single orbit in a plane through the nucleus (at least for elements of high atomic numbers). The revolution in opposite directions has the advantage, among others, of reducing the magnetic field due to the electrons for points at a distance from the atom to a very small value.

The theory contains three fundamental laws. The acceleration law, the angular momentum law and the frequency law. The acceleration law states that the centripetal acceleration of each electron revolving in its orbit equals the centripetal force acting on it, due to the attraction and repulsion of all the electrical charges in the atom acting according to Coulomb's inverse square law. The angular momentum law states that the angular momentum of each electron equals a whole number (called the quantum number), τ, multiplied by Planck's action constant, h, and divided by 2π. According to the frequency law, the product of h into the frequency of vibration, ν, of the radiation emitted during a shift of the electrons from one position of dynamic equilibrium to another equals the difference in the amounts of energy in the atom before and after the shift.

The first two laws cannot be true at every instant of time. One or both of them must represent average values. In the modern development of the theory a definite integral of certain generalized coördinates is equated to a multiple of h.

The theory does not determine the numbers of electrons in the various orbits. In making calculations, however, we must know how the electrons are distributed. Several authors have calculated X-ray frequencies by choosing distributions of electrons in the orbits that best fit the X-ray data themselves. I have taken a distribution suggested by the intervals between the inert gases in the sequence of chemical elements. It has long been supposed that these intervals correspond to groups of electrons in the atom that are completely filled up. From this point of view we get as the numbers of electrons in the various groups the following: the inner orbit contains two electrons. The next group consists of a pair of parallel orbits containing in all eight electrons, four in each orbit. The third group contains eight electrons, four in each of the two parallel orbits. The next group contains eighteen electrons in all, nine in each of the two parallel orbits. The fifth group also contains eighteen, nine in each orbit. The outside pair of parallel orbits contains thirty-two electrons, sixteen in each orbit. Of course, for atoms containing less than ninety-two electrons the outer groups are incomplete, or missing altogether.

To each pair of orbits belongs a certain quantum number, τ. For the inner orbit $\tau = 1$, for the second group $\tau = 2$, for the third group $\tau = 2$, for the fourth and fifth $\tau = 3$, and for the sixth group $\tau = 4$. According to this distribution the number of electrons in an orbit (except the innermost) equals the square of the orbit's quantum number.

The fundamental laws of the theory may be expressed by the following equations:

Acceleration Law.......... $mv^2/a =$ Centripetal Force (1)
Angular Momentum Law.................. $mva = h/2\pi$ (2)
Frequency Law........................ $h\nu = W_1 - W_2$ (3)
Distribution Law............................ $n = \tau^2$ (4)

where m is the mass of an electron, v, its velocity, a, the radius of its orbit, and W_1 and W_2 are the total amounts of energy possessed by the atom in the two states of dynamic equilibrium.

In computing vibration frequencies we first calculate the velocity of each electron from equations 1 and 2, and then find the total energy of all the electrons, including both the potential and kinetic energies. Taking into account the change of mass of an electron with its velocity, the total energy may be represented by

$$W = \Sigma\, m_o c^2(\sqrt{1-\beta^2} - 1),$$

where $\beta = v/c$, c being the velocity of light. If the velocity of the electron is small, as compared with the velocity of light, this expression reduces to

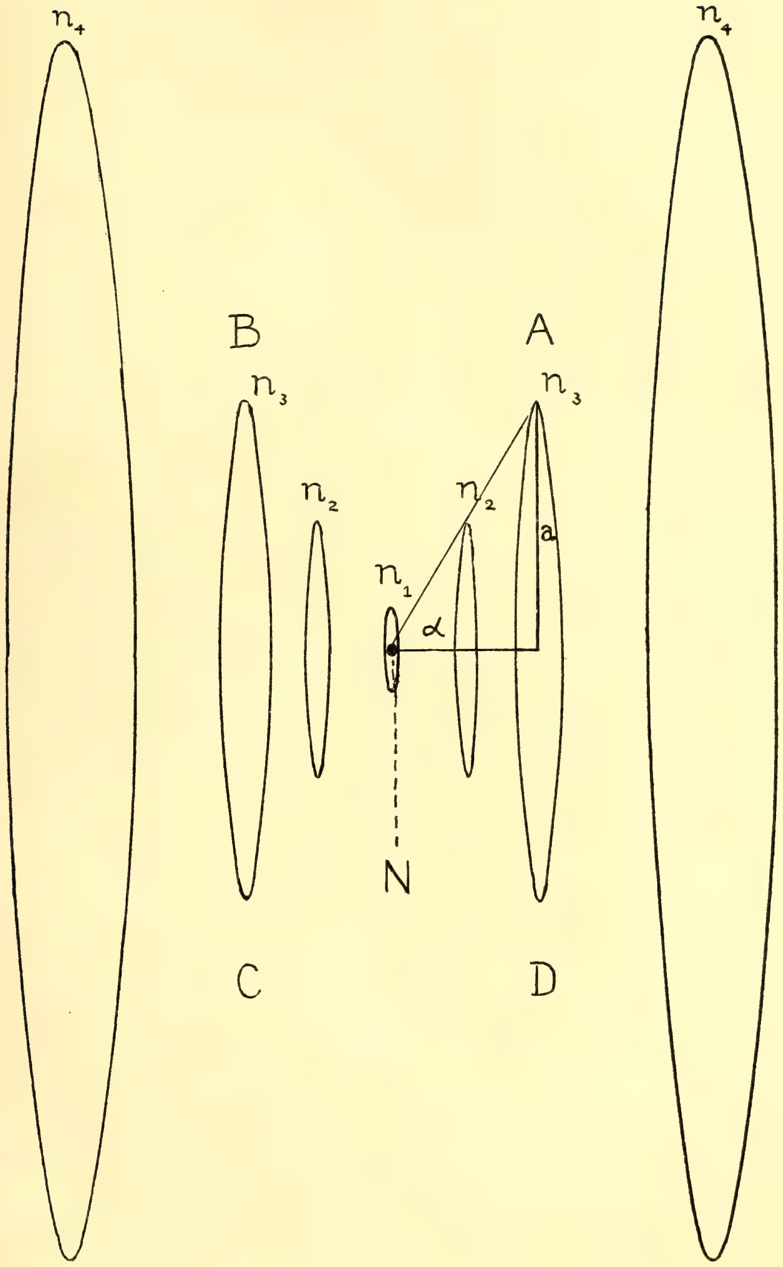
$$W = -\Sigma {}^1/_2\, m_o v^2.$$

To compute the K critical absorption frequency (according to equation 3) we must calculate the value of W for the atom, when one electron is missing from the inner orbit, and subtract from it the value of W, when this electron is in its proper place. The difference between these two terms equals h multiplied by the critical absorption frequency, ν.

Each of the terms in the expressions obtained for W contains the quantity

$$\nu_o = \frac{2\pi^2 e^4 m_o}{h^3} \qquad (5)$$

as a factor. This is the Rydberg fundamental constant. It simplifies the equations to calculate the ratio of the absorption frequency to this fundamental constant, i.e., ν/ν_o.


n_4
n_4
B
A
n_3
n_3
n_2
n_2
a
n_1
α
N
C
D

The terms in the expression for the frequency ratio ν/ν_0 due to the electrons in the inner orbit only are

$$2\,(N-.25)^2\,(1+{}^1/_4\,\beta^2+{}^1/_8\,\beta^4+..)-N^2(1+{}^1/_4\,\beta'^2+\beta'^4+..), \quad (6)$$

where

$$\beta = \frac{2\pi e^2(N-.25)}{ch} \qquad \beta' = \frac{2\pi e^2 N}{ch}$$

and N is the atomic number of the chemical element. In this equation the relatively small forces due to electrons in orbits outside of the inner orbit have been neglected. The two series in increasing powers of β represent the correction for the change of mass of the electron with its velocity.

In calculating the velocities of the electrons in the orbits outside of the inner one, I have made several approximations. Firstly, I have assumed that the force acting on an electron, due to the electrons in orbits that are smaller than its own, is the same as it would be, if these electrons were concentrated at the nucleus of the atom. Secondly, I have neglected the part of the force acting on an electron, due to the electrons in orbits that are larger than its own. Thirdly, the calculation of the force acting on an electron in an orbit, say at A in the figure, due to the electrons in the other orbit BC of the pair I have made by means of two distinct approximations. In the first of these I have assumed that this force equals what it would be, if half the electrons in the orbit BC were concentrated at B, the nearest point in it to A, and half at C, the furthest point from A. This gives only a rough estimate of the terms in the equation representing the ratio ν/ν_0 due to the electrons in the various orbits outside of the inner one. As, however, all these terms add up to only 15 or 20 per cent of the value of the term due to the innermost orbit (expression 6), the error thereby introduced does not appear to be enormous. In the second approximate computation of the force acting on the electron at A, due to the electrons in the orbit BC, I have assumed the electricity of the electrons in BC to be uniformly distributed along the orbit. The results of these computations will be given in a subsequent note. They do not differ much from those obtained from the first approximate computation.

If we calculate the radii of the orbits, we find that, in general, the distances between the electrons in a pair of orbits and the electrons in another pair of orbits are greater in comparison with those radii than as represented in the figure. Hence the influence of electrons in outer orbits is small, and that of electrons in inner orbits approximates to what it would be, if they were at the nucleus.

To calculate the velocity of an electron, say at A, we first calculate the angle α in the figure as follows: Let N' be the total number of electron charges inside the orbits $A\ B\ C\ D$. N' then equals the atomic number of the chemical element N less the number of electrons inside the orbits $A\ B\ C\ D$. Assuming that half the charge in the orbit BC is concentrated at B and the other half at C, equating to zero the horizontal components of the forces due to the nuclear charge and to the electrons in and inside of the orbits $A\ B\ C\ D$ and reducing we get the equation

$$cos^3\alpha = \frac{1}{\frac{8}{n}N' - 1}, \qquad (7)$$

which gives us the value of the angle α.

Equating the radial components of the forces acting on an electron at A to the centripetal acceleration, multiplying by a^2, and substituting the value of N' from equation 7, we get the equation

$$mv^2a = e^2\left(\frac{n}{8}tan^3\alpha - s_n\right) \qquad (8)$$

where

$$s_n = {}^1/_4 \sum_{s=1}^{s=n-1} cosec\, \frac{s}{n} \qquad (9)$$

The velocities of the electrons other than the two in the inner orbit are small as compared with the velocity of light, and the correction for the change of the mass of an electron with its velocity is, therefore, negligible. Neglecting this change and combining equation 8 with the angular momentum law (equation 2) we get for the kinetic energy of the electron

$$^1/_2\, mv^2 = \frac{2\pi^2e^4m}{h^2\tau^2}\left(\frac{n}{8}\, tan^3\alpha - s_n\right)^2 \qquad (10)$$

where the angle α is given by equation 7, and s_n, by equation 9.

In systems of the kind considered, the potential energy of the electrons equals twice their kinetic energy with the negative sign before it.[3] Hence, the total energy, kinetic plus potential, is minus the kinetic energy. Substituting the expression for the total energy of all the electrons with one electron removed from the inner orbit, and with this electron in place in equation 3, and dividing by $h\nu_o$, we get

$$\frac{\nu}{\nu_o} = 2(-N.25)^2(1+{}^1/_4\beta^2+{}^1/_8\beta^4+\ldots) - N^2(1+{}^1/_4\beta'^2+{}^1/_8\beta'^4+\ldots)$$

$$+ \Sigma\, \frac{2n}{\tau^2}\left(\frac{n}{8}\, tan^3\alpha - s_n\right)^2 - {}_n\Sigma\, \frac{2n}{\tau^2}\left(\frac{n}{8} tan^3\alpha' - s_n\right)^2 \qquad (11)$$

as the expression for the critical absorption frequency of the chemical element divided by the Rydberg constant.

Table 1 contains the calculated and the observed values of the ratio ν/ν_0. The observed values I have taken from table 2 of the report on Data Relating to X-Ray Spectra which I compiled for the National Research Council. Column 4 in the table contains the observed values, and column 3 the values calculated according to the following scheme:

$$n_1 = 2, \quad n_2 = 4, \quad n_3 = 4, \quad n_4 = 9, \quad n_5 = 9, \quad n_6 = 16,$$
$$\tau_1 = 1, \quad \tau_2 = 2, \quad \tau_3 = 2, \quad \tau_4 = 3, \quad \tau_5 = 3, \quad \tau_6 = 4.$$

TABLE 1

K CRITICAL ABSORPTION FREQUENCIES

CHEMICAL ELEMENT	ATOMIC NUMBER	FREQUENCIES DIVIDED BY RYDBERG CONSTANT ν/ν_0		
		Calculated	Observed	Calculated
Magnesium	12	103.2	95.8	92.5
Sulphur	16	187.4	181.8	173.3
Calcium	20	297.7	297.5	279.7
Iron	26	521.0	523.8	494.9
Selenium	34	923.3	930.8	888.7
Molybdenum	42	1454.0	1474.0	1411.0
Tin	50	2125.0	2148.0	2063.0
Cerium	58	2918.0	2970.0	2860.0
Dysprosium	66	3896.0	3948.0	3820.0
Tungsten	74	5043.0	5118.0	4954.0
Lead	82	6404.0	6463.0	6270.0
Uranium	92	8468.0	8477.0	8345.0

It appears that, except in the case of the elements magnesium and sulphur, the calculated values in column three do not differ from the observed values by as much as two per cent. Considering the fact that the equations contain no undetermined constants after the distribution law has been fixed and considering the fact that various influences have been neglected, such as magnetic forces, forces due to electrons in outer orbits, the possible influence of electrons which may be forming bonds with other atoms, etc., the close agreement between the calculated and observed values appears extraordinary.

I have also calculated the absorption frequencies, assuming a distribution according to the scheme

$$n_1 = 2, \quad n_2 = 8, \quad n_3 = 18, \quad n_4 = 32,$$
$$\tau_1 = 1, \quad \tau_2 = 2, \quad \tau_3 = 3, \quad \tau_4 = 4.$$

The calculated values based on this scheme do not differ very much from those contained in column 3 of the table. It appears, therefore, that the calculation of the K critical absorption frequencies does not furnish a very sensitive method of determining exactly what the distribution of electrons is. It is important to note, however, that two electrons have a quantum number of one, sixteen electrons have a quantum number of two, thirty-six electrons have a quantum number of three, etc.

On examining the data closely we find a systematic deviation of the calculated from the observed values. This deviation becomes most marked in the elements of low atomic number. It lies in the same direction as the well-known difference between the observed and calculated ionizing potentials of helium.

In order to see whether a better representation of facts can be obtained by supposing that the two inner electrons revolve in opposite directions in separate orbits, I have made the calculations on that basis. The only difference in formula 11 occurs in the terms representing the innermost orbit. The first two terms of equation 11 must be replaced by the single term.

$$(N-1)^2\,(1 + {}^1/_4\,\beta'^2 + {}^1/_8\,\beta'^4 + \ldots) \qquad (12)$$

Column 5 of the table contains the results of the computations. It appears from the data of column 5 that the observed and calculated values differ from each other by amounts up to $4^1/_2\%$. A systematic variation exists which increases as the atomic weight decreases, and which lies in the opposite direction to that represented by the data of column 3.

[1] *Science*, May 21, 1920.

[2] Fall Meeting of the National Academy of Sciences, 1920; *Physic. Rev.*, March, 1921, p. 431.

[3] This can be proved easily for the particular case in question. For a proof of the theorem in a more general form see A. Sommerfeld, *Atombau und Spektrallinien*, Appendix 5.

ON THE CALCULATION OF THE X-RAY ABSORPTION FREQUENCIES OF THE CHEMICAL ELEMENTS (SECOND NOTE)

BY WILLIAM DUANE

JEFFERSON PHYSICAL LABORATORY, HARVARD UNIVERSITY

Communicated July 25, 1921

In a note presented to the National Academy of Sciences[1] I have given some computations of the K critical absorption frequencies of the chemical elements based on the Rutherford-Bohr theory of the structure of atoms and the mechanism of radiation. In these computations I have assumed that the electrons were distributed in circular orbits, which did not lie in planes passing through the nucleus of the atom.

In order to estimate roughly the forces exerted on an electron in one orbit (A in the figure of the previous note) due to the electrons in the parallel orbit, I assumed that they were the same as if the charges of the electrons in the orbit BC were concentrated, half at the nearest point in the orbit to A, and half at the furthest point in the orbit from A. In the computations presented in this note I assume that the forces are the same as if the electricity of the electrons in the orbit BC were uniformly distributed along the orbit. As in the previous note, I neglect the forces acting on an electron at A, due to the electrons in the orbits that are larger than the orbits $A\ B\ C\ D$, and assume that the forces due to the electrons in the orbits smaller than $A\ B\ C\ D$ are the same as if these electrons were concentrated at the nucleus of the atom.

The terms in the expression for the ratio of the critical absorption frequency to the Rydberg constant, ν/ν_0, due to the two innermost electrons, are the same as in the previous case (expressions 6 and 12 of the previous note). To get the correction term due to the electrons in the orbits outside of the inner orbit we proceed as follows: Equating to zero the horizontal components of the forces acting on an electron at A due to the nucleus and to the electrons inside and in the orbits $A\ B\ C\ D$, we get the equation

$$N' cos^3\alpha = \frac{n}{2\pi}\int_0^{\frac{\pi}{2}} \frac{d\theta}{(1 + tan^2\alpha\ sin^2\theta)^{3/2}}$$

where θ is half the angle made by the radius of the orbit BC drawn to any point in the orbit with the vertical.

Similarly, equating the centripetal acceleration of the electron at A to the centripetal force acting on it, due to the nucleus, to the electrons inside the orbits $A\ B\ C\ D$ and to the electricity of the orbit BC, and reducing, we get the equation

$$mv^2a = N' sin^3\alpha - nB - s_n, \tag{14}$$

$$\text{where } B = \frac{tan^3\alpha}{2\pi}\int_0^{\frac{\pi}{2}} \frac{sin^2\theta\, d\theta}{(1 + tan^2\alpha\, sin^2\theta)^{3/2}} \tag{15}$$

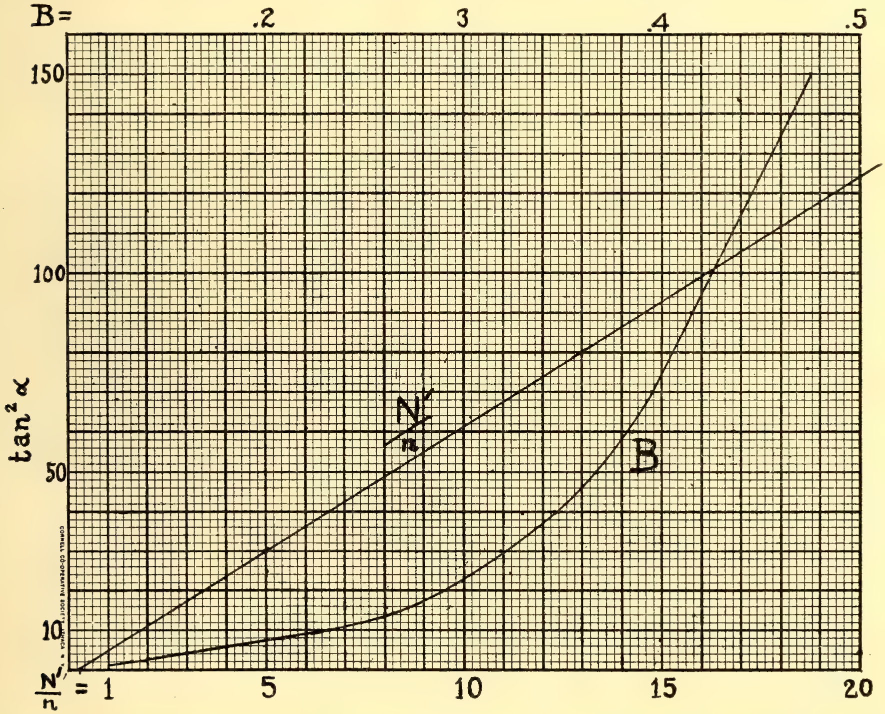
and where s_n is given by equation 9 of the previous note.

Combining equation 14 with equation 2 of the previous note, representing the angular momentum law, we get an expression similar to equation 10 for the kinetic energy of the electron at A. Taking the sum of these expressions for all the electrons with one of the electrons of the inner orbit removed and with it in place, and substituting in formula 3, representing the frequency law, we get the equation for ν/ν_0

$$\frac{\nu}{\nu_0} = 2(N - .25)^2(1 + {}^1/_2\,\beta^2 + {}^1/_8\,\beta^4 + ..) - N^2(1 + {}^1/_4\,\beta'^2 + {}^1/_8\,\beta^{14} + ..)$$
$$+ \sum \frac{2n}{\tau^2}\,[N' sin^3\alpha - nB - s_n]^2$$

$$- \sum \frac{2n}{\tau^2} [(N' + 1)sin^3\alpha' - nB' - s_n]^2 \tag{16}$$

where α is related to N' and n according to equation 13, and B is given by equation 15, and where α' and B' are obtained from the same equations by substituting $N' + 1$ for N'. For purposes of computation I have performed the indicated integrations by graphical methods and plotted curves representing N'/n and B as functions of $tan^2\alpha$. The curves appear in the figure. The relation of N'/n to $tan^2\alpha$ is almost linear. By using these curves to determine the values of α and α' corresponding to various values of N'/n and $(N'+1)/n$, respectively, and then the values of B and B' corresponding to those values of α and α', the data of table 2 has been computed. Columns 3 and 5 in table 2 contain values computed on a distribution of electrons represented by the scheme



$n_1 = 1$, $n_2 = 4$, $n_3 = 4$, $n_4 = 9$, $n_5 = 9$, $n_6 = 16$,
$\tau_1 = 1$, $\tau_2 = 2$, $\tau_3 = 2$, $\tau_4 = 3$, $\tau_5 = 3$, $\tau_6 = 4$.

The data in column 5 has been computed under the assumption that the two electrons nearest the nucleus revolve in opposite directions in separate orbits, one outside the other; whereas for the data of column 3 they are supposed to revolve in the same orbit.

TABLE 2

K Critical Absorption Frequencies

CHEMICAL ELEMENT	ATOMIC NUMBER	FREQUENCIES DIVIDED BY RYDBERG CONSTANT, ν/ν_0		
		Calculated	Observed	Calculated
Magnesium	12	103.3	95.8	92.6
Sulphur	16	181.1	181.8	167.0
Calcium	20	290.0	297.5	272.0
Iron	26	510.1	523.8	484.0
Selenium	34	918.4	930.8	873.8
Molybdenum	42	1431.	1474.	1388.
Tin	50	2056.	2148.	1994.
Cerium	58	2854.	2970.	2796.
Dysprosium	66	3853.	3948.	3777.
Tungsten	74	4990.	5118.	4901.
Lead	82	6362.	6463.	6228.
Thorium	90	7891.	8075.	7774.
Uranium	92	8313.	8477.	8190.

It appears from a comparison of table 2 with table 1 of the previous note that the values computed according to the integral equations are somewhat smaller than those computed by the rough formulas. They differ, therefore, from the observed values more than do the rough calculations. The formula, however, gives the right order of magnitude for the *K* critical absorption frequencies.

In making the computations by the formulas of this and of the preceding note no account has been taken of the influence of electrons in orbits larger than that containing the electron under discussion at *A*. It is quite possible that the forces due to the electrons in these orbits would push the electron at *A* further from the meridian plane of the atom. This would reduce the value of the angle α, reduce the value of the correction term that is subtracted from the main term in the expression for ν/ν_0, and therefore increase the calculated value of ν/ν_0. This might bring the computed values closer to the observed. An accurate estimate of the correction, however, would be extremely difficult to obtain.

If we calculate the radius of an orbit from the above equations, we find that it contains as a factor the square of the quantum number of the orbit, as well as other quantities. This means that the orbits having the same quantum number do not differ very much in radii, but that when the quantum number changes there is a considerable difference in radii. In the above distribution of electrons four orbits have quantum numbers of two, and four have quantum numbers of three. The radii of the two quantum orbits do not differ very much from each other, and the radii of the three quantum orbits do not differ very much from each other, but the latter are much larger than the former. It follows from this that the distribution of electrons described by the above equations consists of two electrons in an inner orbit: then two groups of eight electrons, each, each group on the surface of a sphere, the two spheres lying fairly close together. Considerably outside of these spheres are two spherical surfaces, each containing eighteen electrons and lying fairly close together. Thus the distribution corresponds approximately with that described in the Lewis-Langmuir theory of static atoms.

Bohr[2] has recently suggested an arrangement of orbits, one outside the other, that differs slightly from the arrangement adopted in the preceding calculations. He assumes that the three quantum orbit or orbits lie between the two quantum orbits, and that for those elements which have four quantum electrons the four quantum orbits lie between the two three quantum orbits. This arrangement of electrons may be represented by the scheme

$$n_1 = 2, \quad n_2 = 4, \quad n_3 = 9, \quad n_4 = 16, \quad n_5 = 9, \quad n_6 = 4,$$
$$\tau_1 = 1, \quad \tau_2 = 2, \quad \tau_3 = 3, \quad \tau_4 = 4, \quad \tau_5 = 3, \quad \tau_6 = 2.$$

In order to test this distribution of electrons I have calculated the critical absorption frequencies of the chemical elements that lie just above the inert gases in the chemical tables. The critical absorption wave-lengths have not been measured for all the inert gases themselves. In making the computations I have assumed that the distribution of the electrons is the same as that in the inert gases, and that in each case the extra electron has been taken over by the other chemical element which forms the chemical compound used in measuring the critical absorption wave-length. As a matter of fact, it makes very little difference what is done with this additional electron. Its influence on the critical absorption wave-length is, theoretically, very small. This distribution of electrons is the same as that employed above for all chemical elements up to and including argon ($N = 18$). I have, therefore, included in the computations only those chemical elements of higher atomic number than eighteen. In the case of niton ($N = 86$) the chemical element just above it is missing, and I have, therefore, calculated the critical absorption frequency for thorium. The data appear in table 3. As in tables 1 and 2 column 3 contains the data computed on the supposition that the two one quantum electrons revolve in the same orbit, whereas for the data of column 5 they are supposed to revolve in opposite directions in separate orbits, one just outside the other. It appears that the agreement between the calculated and observed values is much better for this distribution of electrons than for the preceding distributions, especially the data of column 3. Owing to the fact that the radius of the first three quantum orbits is much larger than that of the two quantum orbits inside them the correction for the mutual influence of these two and three quantum orbits becomes very small.

TABLE 3

K CRITICAL ABSORPTION FREQUENCIES

CHEMICAL ELEMENT	ATOMIC NUMBER	FREQUENCIES DIVIDED BY RYDBERG CONSTANT, ν/ν_0		
		Calculated	Observed	Calculated
Potassium	19	262.0	266.0	243.7
Rubidium	37	1128.	1120.	1091.
Caesium	55	2649.	2648.	2591.
Thorium	90	8090.	8075.	7962.

In order to make the computations for chemical elements other than those in table 3 it appears to be necessary to make additional assumptions as to just how the elements are built up in increasing values of the atomic numbers. There is also some difficulty about the radii of the various orbits. In general, a two quantum orbit has a much smaller radius than a three quantum orbit.

The formulas of this and the preceding note contain no undetermined constants after the distribution of the electrons has been fixed; i.e., there is nothing in the equations which must be obtained from X-ray measurements themselves. The formulas are functions of numbers and of π except in the correction term for the change of mass of an electron with its velocity. The correction term is the only place where measured quantities appear. These quantities are the electron's charge e, Planck's action constant h, and the velocity of light c, and they may be regarded as having been determined by experiments other than X-ray measurements. In spite of this fact the formulas represent the observed values to within a few per cent. This is fairly close, considering the influences that have been neglected. Among the reactions that have been neglected we may mention the forces due to electrons in orbits outside the one considered, the fact that electrons inside these orbits have been supposed concentrated in the nucleus, forces due to the magnetic fields of the revolving electrons, the influence of electrons that may be out forming bonds with other chemical elements and which may not, therefore, take part in the energy changes, etc.

The fact that the various distributions considered give results that differ from each other by only a few per cent indicates that the calculation of the K critical absorption frequency is not a sensitive method of determining the *exact* distribution of the electrons. Further, other distributions than those considered may furnish more accurate values. The important point is that the distribution of electrons is what may be called the chemical distribution, and that the numbers of electrons in the orbits are related to their quantum numbers according to equation 4.

The calculation of the L and M, etc. critical absorption frequencies presents very great difficulties, for, if we suppose that an electron is removed from the second or third pair of orbits, it leaves this pair of orbits unbalanced. Just what would happen in this case is not clear, and it would require an additional assumption in order to complete the calculations. Definite general conditions of the dynamic equilibrium have not yet been found.

It may be, also, that orbits that are not circular would give better values than circular orbits. Computations of the frequencies on this basis present formidable difficulties. The fact, however, that the two quantum and three quantum orbits lie not in a plane, but in space of three dimensions may explain the appearance of three critical absorption wave-lengths in the L series, and six critical absorption wave-lengths in the M series, etc.

According to Sommerfeld's theory[3] the difference between two L absorption frequencies is due to the difference in shape of a circular and an elliptic orbit. His formula contains an undetermined constant. Professor Patterson and I have shown[4] that if we assume four electrons in the L orbit the undetermined constant is done away with, and that Sommerfeld's formula represents roughly the difference between the L_1 and L_2 absorption frequencies. It may be that a formula calculated on the basis of three dimensional orbits would give more accurate results.

I am greatly indebted to several of my assistants for carrying through many of the computations.

[1] These PROCEEDINGS, Sept., 1921, p. 260.

[2] *Nature,* March 24, 1921.

[3] *Atombau and Spektrallinien,* Chapter 5.

[4] These PROCEEDINGS, Sept., 1920, p. 517.

SEMI-COVARIANTS OF A GENERAL SYSTEM OF LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS

BY E. B. STOUFFER

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KANSAS

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It is known[1] that the most general transformation of the dependent variables which converts the system of linear homogeneous differential equations

$$y_i^{(m)} + \sum_{l=o}^{m-1} \sum_{k=1}^{n} \binom{m}{l} p_{ikl}\, y_k^{(l)} = 0, \ (i = 1, 2, \ldots, n), \qquad (A)$$

where p_{ikl} are functions of the independent variable x, into another system of the same form is given by the equations

$$y_k = \sum_{\lambda=1}^{n} \alpha_{k\lambda}\bar{y}_\lambda, \quad (k = 1, 2. \ldots, n), \tag{1}$$

where $\alpha_{k\lambda}$ are arbitrary functions of x and where the determinant Δ of the transformation does not vanish identically.

A function of the coefficients of (A) and their derivatives and of the dependent variables and their derivatives which has the same value for (A) as for any system derived from (A) by the transformation (1) is called a *semi-covariant*. If a semi-covariant does not contain the dependent variables or their derivatives, it is called a *seminvariant*. A complete system of seminvariants of system (A) has been calculated.[2] It is the purpose of this paper to obtain the additional semi-covariants necessary for a complete system of semi-covariants.

If equations (1) are solved for $\bar{y}_\lambda$, there results

$$\Delta\bar{y}_\lambda = \sum_{j=1}^{n} A_{j\lambda}\, y_j, \tag{2}$$

where $A_{j\lambda}$ is the algebraic minor of $\alpha_{j\lambda}$ in Δ. If the coefficients in (1) are assumed to satisfy the conditions for the transformation of (A) into the semi-canonical form,[2] the successive differentiation of (2) gives

$$\Delta\bar{y}_\lambda{}^{(\tau)} = \sum_{j=1}^{n} A_{j\lambda}\, t_{j\tau}, \; (\tau = 1, 2, \ldots.), \tag{3}$$

where

$$t_{j\tau} = t'_{j,\,\tau-1} + \sum_{k=1}^{n} p_{j,\,k,\,m-1}\, t_{k,\,\tau-1}, \; t_{jo} = y_j \tag{4}$$

The most general form of (1) which leaves the semi-canonical from in the semi-canonical form is given by the equations[2]

$$y_k = \sum_{\lambda=1}^{n} a_{k\lambda}\, \bar{y}_\lambda, \tag{5}$$

where $a_{k\lambda}$ are arbitrary constants whose determinant D is not zero. The semi-covariants in their semi-canonical form are obtained by transforming the semi-canonical form of (A) by (5). We shall let π_{ikl} denote the coefficients of the semi-canonical form of (A) which correspond to the coefficients p_{ikl} of (A). The effect of the transformation (5) upon $\pi^{(\tau)}_{ikl}$ is given by the equations[2]

$$D\bar{\pi}^{(\tau)}_{ikl} = \sum_{\lambda=1}^{n} \sum_{\mu=1}^{n} A_{\lambda i}\, \pi^{\tau}_{\lambda\mu l}\, a_{\mu k} \ (l=0, 1, \ldots m-2) \qquad (6)$$

where $A_{\lambda i}$ is the algebraic minor of $a_{\lambda i}$ in D.

If we put

$$r_i = \sum_{j=1}^{n} \pi_{i,j,m-2}\, r_{j,l-1} \ (i = 1, 2, \ldots n, l = 1, 2, \ldots n-1), \qquad (7)$$

where $r_{jo} = y_j$, it is easily verified that each of the sets of quantities $r_{il} (i=1, 2, \ldots, n)$ is transformed by (5) cogrediently with y_i $(i=1, 2, \ldots, n)$. Therefore, the determinant

$$R = \begin{vmatrix} y_1, & y_2 & \ldots\ldots & y_n \\ r_{11}, & r_{21} & \ldots\ldots & r_{n1} \\ \ldots & \ldots & \ldots\ldots & \ldots \\ r_{1,n-1} & r_{2,n-1} & \ldots\ldots & r_{n,n-1} \end{vmatrix}$$

is a relative semi-covariant.

Again, it is evident from (6) that

$$s_i = \sum_{j=1}^{n} \pi'_{i,j,m-2}\, y_j, \ (i=1, 2, \ldots n),$$

is a set of quantities which are transformed by (5) cogrediently with y_i $(i=1, 2, \ldots, n)$. We therefore have $n - 1$ additional relative semi-covariants

$$S_i = \sum_{j=1}^{n} s_j \frac{\partial R}{\partial r_{ji}}, \ (i=1, 2, \ldots n-1).$$

Since the coefficients in (5) are constants, each set $y_i^{(\tau)}$ $(i=1,2,\ldots,n)$ of derivatives of y_i are transformed by (5) cogrediently with y_i $(i=1,2,\ldots,n)$. We therefore have $mn - n$ relative semi-covariants

$$T_{l\tau} = \sum_{j=1}^{n} y_j^{(\tau)} \frac{\partial R}{\partial r_{jl}}, \ (l=0, 1, \ldots n-1; \tau=1, 2, \ldots, m-1)$$

A comparison of (3) with the inverse of (5) and of the expressions,[2] π_{ikl}, and π'_{ikl}, in terms of the coefficients of (A), with (6) shows that the semi-covariants R, S_i, $T_{l\tau}$ may be expressed as semi-covariants of (A) simply by replacing $y_i^{(\tau)}$ by $t_{i\tau}$, π_{ikl} by u_{ikl}, and π_{ikl} by v_{ikl}, where u_{ikl} and v_{ikl} are functions of the coefficients of (A) and their derivatives which appear in the expressions for π_{ikl} and π'_{ikl}.

If the transformation (1) and the corresponding transformations for the derivatives of y_i are made infinitesimal, and the resulting system of partial differential equations for the semi-covariants is set up, it is found that there are exactly mn relative semi-covariants which are not seminvariants. We thus have the proper number of semi-covariants, but it remains to show that they are independent.

A comparison of R and S_i with the corresponding semi-covariants[3] for the special case of (A) where $m = 2$ shows R and S_i to be independent. Again, the functional determinant of $T_{l\tau}$ with respect to $y_i^{(\tau)}$ $(i=1,2,\ldots,n)$ for each value of $\tau = 1, 2, \ldots\ldots, m-1$ shows[3] that $T_{l\tau}$ are independent, among themselves and of R, of S_i and of the seminvariants.

We have now proved the following theorem:

All semi-covariants are functions of seminvariants and of R, S_i $(i = 1$ $2, \ldots\ldots\ldots, n-1)$, $T_{l\tau}$ $(l = 0, 1, \ldots\ldots\ldots, n-1;\ \tau = 1, 2, \ldots\ldots\ldots m-1)$.

[1] Wilczynski, E. J., *Projective Differential Geometry of Curves and Ruled Surfaces*, Teubner, Leipzig, Chap. I.

[2] Stouffer, these PROCEEDINGS, **6**, 1920 (645–8).

[3] Stouffer, *London, Proc. Math. Soc.*, (Ser. 2), **17**, 1919 (337–52).

AN ALGORISM FOR DIFFERENTIAL INVARIANT THEORY

By OLIVER E. GLENN

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PENNSYLVANIA

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1. Comprehensive as the existent theory of differential parameters is, as related to quantics

$$F = (a_0, a_1, \ldots, a_m)\,(dx_1, dx_2)^m \quad (aj = aj(x_1, x_2)),$$

under arbitrary functional transformations

$$(1) \qquad x_i = x_i\,(y_1, y_2) \quad (i = 1, 2),$$

developments of novelty relating to the foundations result when emphasis is placed upon the domains within which concomitants of such classes may be reducible, particularly a certain domain $R(I, T, \Delta)$ defined in part by certain irrational expressions in the derivatives of the arbitrary functions occurring in the transformations. For a given set of forms F all differential parameters previously known are functions in R of certain elementary invariants, which we designate as invariant elements, and their derivatives. The theory of invariant elements serves, therefore, to unify known theories and, for the various categories of parameters, gives a means of classification.

It leads also to systems of certain new types which I designate as orthogonal[1] and the extended orthogonal types of differential parameters, and methods of enumeration relating to these systems, both general and particular.

In the paper of which the present article is an abstract, references to the literature are only incidental to the developments but mention may be given to memoirs by Christoffel,[2] Ricci and Levi-Civita,[3] to the tract of J. E. Wright,[4] containing bibliography, and to the symbolical theory of Maschke.[5]

2. The poles of the transformations on the differentials,

$$T : dx_i = \frac{\partial x_i}{\partial y_1} dy_1 + \frac{\partial x_i}{\partial y_2} dy_2 \ (i = 1, 2),$$

are the roots of the linear forms

$$h_{\pm 1} = 2 \frac{\partial x_2}{\partial y_1} dx_1 + \left(\frac{\partial x_2}{\partial y_2} - \frac{\partial x_1}{\partial y_1} \pm \Delta \right) dx_2, \tag{2}$$

where

$$\Delta = \left[\left(\frac{\partial x_2}{\partial y_2} - \frac{\partial x_1}{\partial y_1} \right)^2 + 4 \frac{\partial x_1}{\partial y_2} \frac{\partial x_2}{\partial y_1} \right]^{\frac{1}{2}},$$

and we may place $h_{\pm 1}$ equal to exact differentials

$$\mathrm{T} : df_{+1} = h_{+1}, \ df_{-1} = h_{-1},$$

thus obtaining another transformation upon the differentials of the same degree of arbitrariness as T, since the functional determinant of f_{+1}, f_{-1} will not vanish when $\Delta \neq 0$. The quantics $df_{\pm 1}$ are relative differential covariants appertaining to a domain $R(1, T, \Delta)$ whose defining quantities are the y_1, y_2 derivatives of x_1 and x_2, and the expression Δ. The covariant relations are

$$df'_{+1} = \rho_{+1}^{-1} df_{+1}, \ df_{-1} = \rho_{-1}^{-1} df_{-1}, \tag{3}$$

where

$$\rho_{\pm 1} = \frac{1}{2} \left(\frac{\partial x_1}{\partial y_1} + \frac{\partial x_2}{\partial y_2} \pm \Delta \right),$$

are the factors in $R(1, T, \Delta)$ of the determinant D of T.

3. Let

$$\partial x_1/\partial y_1 = \alpha_1, \partial x_1/\partial y_2 = \alpha_2, \partial x_2 \partial/y_1 = \beta_0, \partial x_2/\partial y_2 = \beta_1,$$
$$\partial f/\partial x_i = f_i \ (i = 1, 2),$$

and write, after Maschke, the quantic F as the m-th power of a symbolical exact differential,

$$F = (df)^m = (f_1 dx_1 + f_2 dx_2)^m = (d\varphi)^m.$$

Then when F is transformed by the inverse of T, and the result expanded, the coefficients φ_{m-2i} $(i = 0, \ldots\ldots, m)$ of the terms in df_{+1}, df_{-1} are differential parameters of the domain $R(1, T, \Delta)$, forming, with $df_{\pm 1}$, a complete system in this domain. Their explicit form is

$$\varphi_{m-2i}=[(\gamma_1-\Delta)f_1-2\beta_0 f_2]^{m-i}[-(\gamma_1+\Delta)f_1+2\beta_0 f_2]^{i}(-4\beta_0\Delta)]^{-m}\ (i=o,..,m)$$

and the invariant relations are

$$\varphi'_{m-2i} = \rho_{+1}^{m-2i}D^{i}\varphi_{m-2i}\ (i=o,\ \ldots,\ m). \tag{4}$$

The functions φ_{m-2i}, df_{+1}, df_{-1} are the invariant elements.

4. By differentiation of the equations (4) relations are obtained as follows:

$$\frac{\partial^r\varphi'_{m-2i}}{\partial y_1^{r-s}\partial y_2^{s}} = \alpha_1^{r-s}\beta_1^{s}\rho_{+1}^{m-2i}D^{i}\frac{\partial^r\varphi_{m-2i}}{\partial x_1^{r-s}\partial x_2^{s}} + \delta_r^{(i)},$$

$$(r = o,\ 1,\ \ldots\ldots,;\ s = o,\ \ldots\ldots,\ r;\ i = o,\ \ldots\ldots,\ m).$$

As a result, employing the transvectant symbol[5]

$$U_1V_2 - U_2V_1 = (U,\ V)$$

we obtain a general category of relative differential parameters,

$$\psi_{m-2i}^{(r)} = (\ldots((\varphi_{m-2i},\ Q^{(i)}),\ Q^{(i)}),\ldots,Q^{(i)})\quad (Q^{(i)}=\rho_{+1}^{m-2i}D^{i}),$$

the number of iterations being r, and

$$\psi_{m-2i}^{(r)'} = \rho_{+1}^{m-2i}D^{i+r}\psi_{m-2i}^{(r)};\ (i=o,\ldots,m). \tag{5}$$

We define $\psi_{m-2i}^{(o)}$ to be φ_{m-2i}.

Write

$$P_{+1} = \prod_{k=o}^{r}\psi_{m}^{(k)X_{ok}}\ \psi_{m-2}^{(k)X_{1k}}\ldots\psi_{-m}^{(k)X_{mk}}df_{+1}^{\sigma_2}df_{-1}^{\sigma_1}$$

and let the conjugate product be P_{-1};

$$P_{-1} = \prod_{k=o}^{r}\psi_{-m}^{(k)X_{ok}}\psi_{-m+2}^{(k)X_{1k}}\ldots\psi_{m}^{(k)X_{mk}}df_{+1}^{\sigma_2}df_{-1}^{\sigma_1},$$

then the parameters of the extended orthogonal type are defined to be those which can be generated in totality by forming such rational expressions in $\psi_{m-2i}^{(k)}$, $df_{\pm 1}$ as simplify by multiplication into functions appertaining to the domain $R(I,T,0)$, free, that is, from the expression Δ. The essential forms from which to construct this totality are evidently $P_{+1} \pm P_{-1}$. When $r = 0$ the type is called orthogonal.[1]

Finite complete systems can be derived in this theory. In fact the products P form a Hilbert system of monomials whence it follows that a complete system of concomitants of the extended orthogonal type is furnished by the finite set of irreducible solutions of the diophantine equation

$$\sum_{k=o}^{r}\ \sum_{l=o}^{m}(m-2l)X_{kl}-\sigma_1+\sigma_2=o. \tag{6}$$

Particular systems of the orthogonal type have been constructed by the present writer in this and previous papers for the quantics of orders one to six inclusive, the system of the sextic for example being composed of 31 parameters. For the case $r = 1$, $m = 2$ the system comprises eleven concomitants of the extended type as follows:

$$\varphi_0,\ \psi_0^{(1)},\ n = \varphi_2\varphi_{-2},\ o = (\varphi_2,\ \rho_{+1}^{2})\ (\varphi_{-2},\ \rho_{-1}^{2}),$$

$$p_{\pm 1} = \varphi_2(\varphi_{-2},\ \rho_{-1}^{2}) \pm \varphi_{-2}(\varphi_2,\ \rho^{2}{}_{+1}),$$

$$q_{\pm 1} = (\varphi_2, \rho^2_{+1})df^2_{+1} \pm (\varphi_{-2},\rho^2_{-1})df^2_{-1},$$
$$r_{\pm 1} = \varphi_2\, df^2_{+1} \pm \varphi_{-2}df^2_{-1},\ s = df_{+1}\, df_{-1}.$$

5. A paper giving the above and other developments in particular is to appear in the September (1921) number of the *Annals of Mathematics*. An article relating to the rôle of invariant elements[6] in algebraic orthogonal and boolean invariant theory appeared in the *Transactions* of the American Mathematical Society, vol. 20 (1919), and a paper containing like theory for invariants of relativity and modular invariants appeared in the same *Transactions* for the year 1920. Further researches are in progress.

[1] Elliott, *Proc. London Math. Soc.*, **33** (1901).

[2] Christoffel, *Crelle, J. Math.*, **70** (1869).

[3] Ricci and Levi-Civita, *Math. Ann.*, **54** (1901).

[4] J. E. Wright, *Invariants of Quadratic Differential Forms*, Cambridge tracts (1908).

[5] Maschke, *Trans. Amer. Math. Soc.*, **1** (1900); **4** (1903).

[6] O. E. Glenn, *Ann. Math.*, **20** (1918).

THE EFFECT OF TEMPERATURE AND OF THE CONCENTRATION OF HYDROGEN IONS UPON THE RATE OF DESTRUCTION OF ANTISCORBUTIC VITAMIN (VITAMIN C)

By H. C. SHERMAN, V K. LA MER AND H. L. CAMPBELL

DEPARTMENT OF CHEMISTRY, COLUMBIA UNIVERSITY

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In these experiments the vitamin was determined by means of feeding experiments with guineapigs according to the general method made familiar by Hess' recent monograph.[1] Cohen and Mendel,[2] Givens and McCluggage,[3] and the workers at the Lister Institute[4] among other investigators of the antiscorbutic vitamin, have emphasized the importance of basal diets which shall supply all other nutritive requirements and yet furnish none of the vitamin in question or only negligible traces of it. Further investigation of this point led the present writers to adopt the following as an improvement upon the diets previously proposed.

Basal Ration Used.—Ground whole oats were mixed with skimmed milk powder which had been heated sufficiently to ensure destruction of such antiscorbutic vitamin as it might contain without so changing the flavor as to cause it to be refused by the experimental animals. The heat treatment necessary to ensure complete destruction of vitamin C in the skimmed milk powder should be determined by each investigator for his own material and technique of heating. In our experiments two hours' heating at 110° C. in shallow trays freely exposed to air was found sufficient. To the mixture of oats and skimmed milk powder were added pure fresh butterfat and sodium chloride. The proportions adopted were:

Ground whole oats	59 per cent
Heated skimmed milk powder	30 per cent
Butterfat	10 per cent
Sodium chloride	1 per cent

By using heated skimmed milk and fresh butterfat instead of heated whole milk, the absence of antiscorbutic vitamin is equally well ensured and the palatability of the diet and its fat-soluble vitamin content are improved. This diet supports excellent growth up to the time of the onset of scurvy.

Experimental Animals.—Our experience with about 200 experimental animals indicates that young guineapigs from six to eight weeks old and weighing 300 to 350 grams are best used for studies of vitamin C. Placed upon the above ration at this age and size they usually eat about 18 to 20 grams per day and continue to grow for about 15 days, then lose weight rapidly and die of scurvy in from 26 to 34 days after being confined to the diet. If much older, the animals are somewhat less susceptible; if much younger, the results are less regular.

Symptoms, Survival Period, and Autopsy Findings.—The first symptoms appear after about 12 days on the basal diet. In our cases on the above basal diet only, the survival periods were, respectively, 33, 26, 27, 28, 34, 28, 27, 34, 26, 29, 32, 32, 32, 31, 34 days. The nature of the symptoms and autopsy findings agreed so closely with the descriptions of Cohen and Mendel and of Hess as not to require further discussion here. At autopsy the findings which proved most significant were: looseness of teeth, fragility of bones, enlargements and hemorrhages of joints and of rib junctions. When the animal receives some antiscorbutic vitamin but not enough for protection from scurvy, life is prolonged, but to a rather uncertain extent, and some of the symptoms and autopsy findings may become even more severe than in the animals that receive no antiscorbutic and die more quickly.

Quantitative Rating.—From what has just been said it necessarily follows that exclusive dependence upon either the survival period or any one set of symptoms or autopsy findings might easily prove misleading. All of these are, therefore, considered in each case. This was done with several series of animals, some receiving the basal ration only, and others measured amounts of filtered canned tomato juice up to the amount which gave complete protection from scurvy. By comparison with the standard protocols thus established, for data of which reference must be made to the fuller account of the work to be published elsewhere, it becomes possible to form a quantitative estimate of the degree of protection (if any) which the diet, alone or with such antiscorbutic addenda as have been introduced, has afforded the animal.

Effect of Heating in Acid Solution.—In the case of tomato juice of natural acidity, pH = 4.3, it was found that boiling for one hour destroyed practically 50%, and for four hours destroyed practically 68% of the antiscorbutic vitamin present. The time curve of the destructive process is, therefore, much flatter than that of a unimolecular reaction or of a reaction proceeding according to the square root rule of Schütz. Similar flattening of the time curves of the destruction of the vitamin were found also in experiments in which the tomato juice was heated at temperatures lower than boiling, viz., at 60° and at 80°. Comparisons of the data obtained at 60°, 80° and 100° show relatively low temperature coefficients: Q_{10} (60°–80°) = 1.23; Q_{10} (80°–100°) = 1.12.

Effect of Reduced Concentration of Hydrogen Ions.—In experiments in which the natural acidity of the standard tomato juice was first neutralized in whole or in part, the juice then boiled for one hour and immediately cooled and reacidified, it was found that at pH 5.1 to 4.9 (natural acidity less than half neutralized) the destruction during one hour's boiling was increased to 58%. Neutralization of a larger proportion of the natural acidity increased the rate of destruction of the vitamin. When alkali was added to an initial pH of 11, which fell to about 9 during the hour of heating, the destruction found by feeding of the juice thus treated but immediately cooled and reacidified, was about 65%.

In all of these experiments the heating was performed in cotton-stoppered, narrow-necked flasks from which air was probably very largely displaced by water vapor early in the heating. When the experiments upon heating at 100° were repeated with oxygen bubbling through the solution the destruction of the vitamin was much more rapid.

Heating at 100° for one hour at pH = 11 to 9 as described above followed by standing for one to five days in stoppered but only partially filled bottles in a refrigerator at 10° C. at an alkalinity of only pH = 9 was found to destroy 90 to 95% of the antiscorbutic vitamin, as compared wth 65 per cent when the solution was re-acidified after heating. This confirms the observations of Harden and of Hess upon the susceptibility of this vitamin to alkalinity even at low temperatures.

Thus while the great instability of the antiscorbutic vitamin makes it an unpromising material for attempts at actual isolation, the development of methods for its quantitative measurement makes possible the study of its chemical behavior.

[1] Hess, A. F., *Scurvy Past and Present,* Philadelphia, 1920.

[2] Cohen, B. and Mendel, L. B., *J. Biol. Chem.*, **35,** 1918 (425).

[3] Givens, M. A. and McCluggage, H. B., *Ibid.*, **37,** 1919 (255).

[4] Chick, H., Hume, E. M., Delf, E. M. and others; A series of papers in *Biochem. J.*, 1918, et seq.

INFORMATION TO CONTRIBUTORS

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PROCEEDINGS
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Volume 7 OCTOBER 15, 1921 Number 10

THE EVALUATION OF QUANTUM INTEGRALS

By Edwin C. Kemble

Jefferson Physical Laboratory, Harvard University

Communicated by E. H. Hall, April 22, 1921

The application of the Wilson-Sommerfeld quantum conditions to a conditionally periodic system with orthogonal coördinates involves the evaluation of an integral of the type

$$J = \oint \sqrt{f(q)}\, dq. \tag{1}$$

The integral is to be extended over a complete cycle of values of q, which oscillates between two roots of $f(q)$. The sign of the radical is to be the same as that of dq, so that if a and b denote the roots of $f(q)$, the integral can be written

$$J = 2\int_a^b \sqrt{f(q)}\, dq \tag{2}$$

If $f(q)$ is a polynomial of the second degree in either q or $1/q$ the integral can be cleanly evaluated. Otherwise, approximations are generally necessary. If $f(q)$ can be expressed in the form

$$f(q) = \varphi(q) + \alpha\,\psi(q)$$

where $\varphi(q)$ is quadratic in q or $1/q$, α is constant, and $\alpha\psi(q)$ is small, a natural method of procedure is to try to develop J into a power series in α. Thus

$$J(\alpha) = J(0) + \alpha J'(0) + \frac{\alpha^2 J''(0)}{2!} + \frac{\alpha^3 J'''(0)}{3!} + \cdots \tag{3}$$

$J(0)$ and $J'(0)$ are easily evaluated, but unfortunately the higher derivatives of J with respect to α cannot be calculated by the usual methods because the higher derivatives of $\sqrt{(fq)}$ with respect to α become infinite at $q = a$ and $q = b$. Hence this method is useful only when the higher order terms are negligible.[1]

Another method of attack employed by F. Tank[2] and accepted as valid by others[3] turns out on close examination to be faulty. Tank develops $f(q)$ into a power series about its maximum point, $q = d$. Let $\xi = q-d$, and let H denote the maximum value of $f(q)$. Then $f(q)$ can be thrown into the form

$$f(q) = H - \alpha\xi^2 - (\beta\xi^3 + \gamma\xi^4 + \delta\xi^5 + \ldots). \tag{4}$$

Introducing the symbols

$$V(\xi) = \alpha\xi^2,$$
$$\Delta V = \beta\xi^3 + \gamma\xi^4 + \delta\xi^5 + \ldots,$$

we obtain

$$\sqrt{f(q)} = \sqrt{H-V}\sqrt{1-\frac{\Delta V}{H-V}} = \sqrt{H-V} - \frac{1}{2}\frac{\Delta V}{\sqrt{H-V}} - \frac{1}{8}\frac{(\Delta V)^2}{\sqrt{(H-V)^3}} - \ldots$$

Tank in effect integrates this series term by term between the limits $-(H/\alpha)^{½}$ and $+(H/\alpha)^{½}$ and identifies twice the sum of the series so obtained with J. This procedure is wrong, since the correct limits of integration for J are $\xi = a - d$ and $\xi = b - d$. Moreover, the expansion is not usually convergent throughout the interval of integration, so that it is not possible to correct Tank's work by altering the limits.

Another scheme of series development may be suggested, which avoids the above difficulty. Let the quantity u be defined by the equation

$$H = \sqrt{H - f(q)} = \xi\sqrt{\alpha + \beta\xi + \gamma\xi^2 + \delta\xi^3 + \ldots.} \tag{5}$$

The sign of u is to be the same as that of ξ. The integral J can now be thrown into the form

$$J = 2\int_{u=-\sqrt{H}}^{u=+\sqrt{H}} \sqrt{H-u^2}\,\frac{d\xi}{du}\,du \tag{6}$$

Let us assume that $d\xi/du$ can be developed into a power series in u. Thus

$$\frac{d\xi}{du} = \sum_{\tau=0}^{\infty} a_\tau u^\tau \tag{7}$$

Let

$$K_\tau = 2\int_{-\sqrt{H}}^{+\sqrt{H}} u^\tau\,\sqrt{H-u^2}\,du. \tag{8}$$

To evaluate K_τ we introduce the variable of integration θ defined by the relation $u = H^{1/2}\sin\theta$.

Then (8) becomes

$$K_\tau = H^{\frac{\tau+2}{2}}\int_0^2 \sin^\tau\theta\,\cos^2\theta\,d\theta. \tag{9}$$

It is easy to show from (9) by the application of well-known formulae that K_τ vanishes for odd values of τ; that K_o is πH; and that for even values of τ greater than zero K_τ is given by the equation

$$K_\tau = 2\pi H^{\frac{\tau+2}{2}} \varphi(\tau) \tag{10}$$

where

$$\varphi(\tau) = \frac{1.3.5.\dots(\tau-1)}{2.4.6.\dots(\tau+2)}. \tag{11}$$

Now (6) becomes

$$J = \pi H\,[a_o + 2\sum_{\tau=1}^{\infty} a_{2\tau}\varphi\,(2\tau)H^\tau]. \tag{12}$$

We proceed to the evaluation of the coefficients $a_0, a_1, a_2, \dots$. As $d\xi/du$ may be regarded as a function of either ξ or u, let

$$d\xi/du = \psi(u) = \chi(\xi).$$

Then

$$a_\tau = \frac{1}{\tau!}\overset{(\tau)}{\psi}(0). \tag{13}$$

Since u and ξ vanish together, the derivatives of ψ at the point $u = 0$ can be calculated from the derivatives of χ at the point $\xi = 0$. The following method of procedure is perhaps the simplest. Let

$$v\,(\xi) = \frac{\mathrm{H}-f(q)}{\xi^2} = \alpha + \beta\xi + \gamma\xi^2 + \dots \tag{14}$$

Then

$$u=\xi\ ^{1/2} \text{ and } \frac{du}{d\xi} = \frac{v + {}^1/_2\,\xi v'}{\sqrt{v}}.$$

Let

$$w = 2\,v(\xi) + \xi\,v'\,(\xi) = -\frac{f'(\xi+d)}{\xi}. \tag{14'}$$

Then

$$\chi(\xi) = \frac{1}{\frac{du}{d\xi}} = \frac{2\sqrt{v(\xi)}}{w\,(\xi)}. \tag{15}$$

Differentiating,

$$\left.\begin{aligned} &\psi'(u) = \chi'\,(\xi)\,\chi\,(\xi) \\ &\dots\dots = \dots\dots\dots\dots \\ &\psi^{(n)}(u) = \chi(\xi)\,\frac{d}{d\xi}\left\{\overset{(n-1)}{\psi}\,(u)\right\} \\ &\dots\dots = \dots\dots\dots\dots\dots \end{aligned}\right\} \tag{16}$$

In some cases these successive derivatives of ψ are simple and easily calculated functions of ξ. In others the successive derivatives rapidly become complicated. If $f(q)$ is given as a power series in ξ the process of differentiation can be performed conveniently as follows:

$$\left.\begin{aligned}
\psi'(u) &= \frac{2v'}{w^2} - \frac{4vw'}{w^3}, \\
\psi''(u) &= \frac{4\sqrt{v}}{w^3}\left\{ v'' - \frac{4v'w'}{w} - \frac{2vw''}{w} + \frac{6v(w')^2}{w^2} \right\} \\
\psi'''(u) &= \frac{4}{w^4}\left\{ v'v'' + 2vv''' - \frac{14vv''w'}{w} - \frac{4(v')^2w'}{w} - \frac{14vv'w''}{w} \right. \\
&\quad \left. + \frac{50vv'(w')^2}{w^2} + \frac{40v^2w'w''}{w^2} - \frac{4v^2w'''}{w^2} - \frac{60v^2(w')^3}{w^3} \right\}.
\end{aligned}\right\} \quad (17)$$

. . . .

At the point

$$\left.\begin{aligned}
&u = \xi = 0, \\
&w = 2v = 2\alpha, \qquad w' = 3v' = 3\beta \\
&w'' = 4v'' = 8\gamma \qquad \ldots\ldots\ldots \\
&w^{(n)} = (n+2)\,v^{(n)} = \ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\} \quad (18)$$

Hence

$$\left.\begin{aligned}
\psi(0) &= \frac{1}{\sqrt{\alpha}} \\
\psi''(0) &= \frac{3}{\alpha^{5/2}}\left\{ \frac{5}{4}\frac{\beta^2}{\alpha} - \gamma \right\}, \\
\psi^{\mathrm{IV}}(0) &= \frac{105}{\alpha^{7/2}}\left\{ \frac{33}{16}\frac{\beta^4}{\alpha^3} - \frac{9}{2}\frac{\beta^2\gamma}{\alpha^2} + \frac{2\beta\delta}{\alpha} + \frac{\gamma^2}{\alpha} - \frac{4}{7}\epsilon \right\}.
\end{aligned}\right\} \quad (19)$$

Combining the above equations with (13), (11), and (12), we obtain

$$\begin{aligned}
J &= \pi\psi(o)H + 2\pi\varphi(2)\frac{\psi''(0)}{2!}H^2 + 24\varphi(4)\frac{\psi^{\mathrm{IV}}(0)}{4!}H^3 + \ldots \\
&= \frac{\pi H}{\sqrt{\alpha}}\left\{ 1 + \frac{3}{8\alpha^2}\left[\frac{5}{4}\frac{\beta^2}{\alpha} - \gamma\right]H + \frac{35}{64\alpha^3}\left[\frac{33\beta^4}{16\alpha^3} - \frac{9}{2}\frac{\beta^2\gamma}{\alpha^2} \right.\right. \qquad (20) \\
&\qquad \left.\left. + \frac{2\beta\delta}{\alpha} + \frac{\gamma^2}{\alpha} - \frac{4}{7}\epsilon\right]H^2 + \ldots\ldots \right\}.
\end{aligned}$$

This result is similar in form to Tank's and the coefficients of the two lowest powers of H are the same as in his series, but the coefficient of H^3 is different.

In order that the result may be valid, it is necessary that the series (4) converge and represent $f(q)$ throughout the interval of integration, and that the derivatives of ψ shall all be finite throughout a circle whose radius is greater than $+H^{1/2}$ drawn about the origin on the complex u plane. From (17) it is evident that the derivatives of ψ will be finite up to the point where w vanishes. It follows from (14′) that $f'(q)/(q-d)$ must have no zeros on that part of the complex q plane which is mapped on the above-mentioned circle on the u plane. It is necessary, in particular, that $f'(q)$ shall vanish not more than once for real values of q between a and b.

It does not seem worth while to attempt an exact discussion of the boundaries of the region on the complex q plane from which the zeros of $f'(q)/(q-d)$ must be excluded, but we can say qualitatively that there is little chance that the series (7) will not converge throughout the interval of integration if the series

$$w = 2\alpha + 3\beta\xi + 4\gamma\xi^2 + 5\delta\xi^3 + \dots \qquad (21)$$

converges rapidly.

In the applications of this analysis an expression for H as a function of J will generally be desired. The power series may be reversed to advantage by the following scheme which resembles that of Lagrange. Let $H = F(J)$. Then by Taylor's theorem

$$H = J\,F'(O) + \frac{J^2}{2!}F''(O) + \frac{J^3}{3!}F'''(O) + \dots. \qquad (22)$$

Let b_n denote the coefficient of H^n in (20). $F'(O)$, $F''(O)$, etc., can be calculated in terms of the b's. Let us first compute the derivatives of F in terms of H. Let

$$p(H) = \sum_{n=1}^{\infty} n b_n H^{n-1} = \frac{dJ}{dH}.$$

Then

$$\frac{dH}{dJ} = F'(J) = \frac{1}{p(H)}. \qquad (23)$$

Differentiating again,

$$\left.\begin{aligned} F''(J) &= \frac{d}{dH}\left(\frac{1}{p}\right)\cdot\frac{dH}{dJ} = -\frac{p'}{p^3} \\ F'''(J) &= \frac{d}{dH}\left(-\frac{p'}{p^3}\right)\frac{dH}{dJ} = -\frac{p''}{p^4} + \frac{3(p')^2}{p^5} \\ &\dots\dots\dots\dots\dots\dots \end{aligned}\right\} (24)$$

Since J vanishes when H does, the values of $F'(O)$, $F''(O)$, etc., are obtained by setting H equal to zero in the right-hand members of (23) and (24). Thus

$$\left.\begin{aligned} &F'(O) = 1/b_1; \qquad F''(O) = -\,2b_2/b_1^3; \\ &F'''(O) = -\,6b_3/b_1^4 + 12b_2^2/b_1^5; \\ &\dots\dots\dots\dots\dots\dots \end{aligned}\right\} (25)$$

If these formulas for $F'(O)$, $F''(O)$, etc., are evaluated in terms of the expressions for the b's given by (20), equation (22) becomes

$$H = \frac{\sqrt{\alpha}}{\pi}J + \frac{3}{8\pi^2\alpha}\left[\gamma - \frac{5\beta^2}{4\alpha}\right]J^2 + \frac{1}{32\pi^3\alpha^{3/2}}\Big[10\epsilon - \frac{35\beta\delta}{\alpha}$$

$$-\frac{17}{2}\frac{\gamma^2}{\alpha}+\frac{225}{4}\frac{\beta^2\gamma}{\alpha^2}-\frac{705}{32}\frac{\beta^4}{\alpha^3}\Big]J^3+\ldots\ldots \tag{26}$$

As a check on the series development here suggested, the writer has derived Sommerfeld's formula,

$$\oint\sqrt{A+\frac{2B}{q}+\frac{C}{q^2}}\,dq = 2\pi\sqrt{-C}\left[\frac{B}{\sqrt{AC}}-1\right] \tag{27}$$

by means of the series (12). The derivatives of ψ when evaluated by equations (16) are in this case particularly simple and as a result the series can be summed.

It is of interest to note that the method of development in series here suggested is applicable to a variety of problems. It may be used to evaluate indefinite as well as definite integrals.

One simple application is in the determination of the periods of oscillation of a vibrating system. In the case of a conditionally periodic system with orthogonal coördinates, the periods are given by expressions of the form

$$T = \oint\frac{dq}{\dot{q}} = 2\int_a^b\frac{dq}{\sqrt{f(q)}}, \tag{28}$$

where a and b are again roots of $f(q)$. Introducing the quantities H, ξ, u defined as in the preceding work, we obtain

$$T = 2\int_{-\sqrt{H}}^{+\sqrt{H}}\frac{d\xi\,/\,du}{\sqrt{H-u^2}}\,du. \tag{29}$$

If $d\xi/du$ is developed into a power series of the type (7), we obtain

$$T = \sum_{\tau=0}^{\infty} a_\tau \overline{K}_\tau$$

where

$$\overline{K}_\tau = 2\int_{-\sqrt{H}}^{+\sqrt{H}}\frac{u^\tau\,du}{\sqrt{H-u^2}} = H^{\tau/2}\int_0^{2\pi}\sin^\tau\theta\,d\theta.$$

The coefficients $\overline{K}_\tau$ vanish when τ is odd, and the final expression for the period is

$$T = 2\pi\left[a_0 + 2\sum_{\tau=1}^{\infty}a_{2\tau}\,(\tau+1)\,\varphi\,(2\tau)\,H^\tau\right]. \tag{30}$$

---

[1] This is in effect the method used by Sommerfeld, *Atombau und Spektrallinien*, 2nd edition, Braunschweig, 1921, pp. 476–482.

[2] F. Tank, *Mitteilungen Physik. Ges. Zurich*, 1919, p. 87.

[3] Cf. G. Hettner, *Zs. Physik.*, **1**, 4, 1920 (350).

THE FURTHER EXTENSION OF THE ULTRA-VIOLET SPECTRUM AND THE PROGRESSION WITH ATOMIC NUMBER OF THE SPECTRA OF LIGHT ELEMENTS

BY R. A. MILLIKAN

RYERSON PHYSICAL LABORATORY, UNIVERSITY OF CHICAGO

Read before the Academy April 26, 1920

The chief purpose of this investigation, outlined and begun in 1916[1] and briefly reported upon last year,[2] has been to explore in the extreme ultra-violet the radiations which can be emitted by the second ring or shell of electrons in the atoms of atomic number from 2 to 13 (helium to aluminium).[3]

The results obtained to date may be very briefly summarized as follows:[3]

1. The ultra-violet spectrum has been photographed and its wave-lengths determined down to $\lambda = 136.6$ Ångstroms in the case of aluminium and down to $\lambda = 149.5$ Ångstroms in the case of copper. There is thus a gap represented by a factor of but 10 between the shortest measured ultra-violet waves and the longest X-rays measured by the method of crystal-spectrometry which stops at 13.3 Ångstroms. *Fortunately, however, in the ultra-violet region, which has been already opened up and explored, the most interesting and the most important of the hitherto inaccessible radiations are found;*—thus

2. The L_α lines of Al, Mg, and Na have been photographed and located at 144.3 Å, 232.2 Å, and 372.2 Å, respectively. These wave-lengths are all fairly accurately on the Mosely line connecting L_α frequencies and atomic number (see fig. 1). *It has thus been definitely proved that the L series continues with its main characteristics unchanged throughout the whole range of atomic numbers from Ur (92) to Ne (10).* The linear progression thus revealed clear down to neon could be very roughly inferred from the beautiful measurements of Hjalmar,[4] which gave the K_α and K_β lines of the elements down to sodium. For the Kossel relation between the frequencies of the K and L series, namely, $K_\beta - K_\alpha = L_\alpha$, although known to be quite inexact, was presumably sufficiently near to the truth to enable *the order of magnitude* of the L_α wave-lengths to be predicted. Kossel has already made this use of Hjalmar's data.[5] His values of L_α computed for Al, Mg and Na in this way are actually about 20% too low.

3. It has been found that *the aluminium atom (atomic number 13) when excited by these condensed sparks in vacuo emits no radiations whatever of wave-length between 144.3 Å and about 1200 Å where its M spectrum,* that due to its three outer electrons, begins and extends with considerable complexity into the visible. *This shows that optical spectra are quite like*

X-ray spectra in that large gaps occur between the frequencies due to the electrons in successive rings or shells. If we could neglect the influence of the negative electrons upon one another we could compute the relative diameters of these shells, for they would be inversely proportional to the limiting frequencies, i.e., in this case in about the ratio 1 to 9; but such computations are of little value save as the roughest sort of indices. The chief lines below 2000 Å, due to the three outer electrons of the Al atom have the wave-lengths 1379.7 Å, 1384.5 Å, 1605.9 Å, 1612.0 Å, 1671.0 Å, 1854.7 Å and 1862.7 Å.

Magnesium shows a behavior quite like that of aluminium in that we find a complete blank between its L_α line at 232.2 Å and the lines due to its two outer electrons whose radiations begin on our plates on the short wave-length side at about 1700 Å. The chief radiations below 2000 Å, arising from these two outer electrons of the magnesium atom, have the wave-lengths 1735.2 Å, 1737.9 Å, 1751.0 Å, and 1753. 7 Å.

Also, *sodium*, quite consistently with the foregoing, is found to emit no radiations whatever between its L lines, the longest of which is at 376.5 Å and the lines due to its single M electron which have their convergence wave-length at 2412.63 Å, and reach their maximum intensity in the familiar sodium doublet at 5890 Å and 5896 Å.

4. Coming now to the group of atoms below neon, atomic number 10, the spectra due to the electrons in the incomplete second or L-ring in the case of these atoms are completely unpredictable from any theory that we now have but they have been experimentally obtained. For reasons which will appear they will not be considered in the order of atomic number.

a. The spectrum due to the six L-ring electrons of the oxygen atom (atomic number 8) begins upon our plates at about 230 Å and extends with much complexity and strength up to 834.0 Å where the strongest oxygen line, which will be arbitrarily called its L_α line, is found. Above 834.0 Å the oxygen lines are few in number and relatively faint.

Since the K_α line of O can be computed with much certainty to be at 23.68 Å, the ratio of the K frequency to the L frequency for the oxygen atom is about 35. It will be recalled that this ratio is only about 7 in the case of atoms of high atomic number and that it slowly increases with decreasing atomic number, reaching the values 17.2, 23.4 and 31.1 in the cases of Al, Mg, and Na, respectively (see above). The strongest of the oxygen lines have the following wave-lengths: 321.2, 374.3, 507.8, 525.7, 554.2, 599.5, 610.1, 616.7, 625.2, 629.6, 644.0, 703.1, 718.5 and 834.0.

The oxygen lines have been identified because they appear as impurities in all easily oxidizable metals, NrLi, Mg, Al, Zn, Fe, etc., The key to the oxygen spectrum was furnished by the discovery that chemically pure aluminium and magnesium showed the extraordinary property of having identical spectra in all particulars between 1000 Å and 235 Å, while showing altogether distinct characteristic spectra between 1700 Å and 6000 Å, a behavior which could only mean that they themselves had no lines in the region, between 235 Å and 1000 Å, all of the observed lines being those of the common impurity oxygen—an interpretation supported by the fact that the well known oxygen lines of the visible region were also found in both spectra. It was with the aid of this discovery and the careful comparison of the spectra of otherwise pure metals, or of metals containing known impurities that the origins of practically all of the hundreds, even thousands, of new lines have been established with but very little uncertainty.

b. The spectrum due to the 4 L-ring electrons of the *carbon* atom begins upon our plates at 360.5 Å and extends with much complexity and strength up to 1335.0 Å where its strongest line, which, in harmony with the foregoing convention will be called its L_α[6] line is found. Above 1335.0 Å, the carbon lines are widely scattered and relatively weak. Since the K line of carbon is accurately computed to be at 44.4 Å, the ratio of the K to the L frequency in carbon is about 30.

It is interesting that practically the whole group of lines which Lyman obtained from his *condensed discharge in helium*[7] are due to carbon, oxygen, nitrogen and hydrogen. Thus those whose wave-lengths he gives as 599.0, 643.7, 702.9, 718.2, 796.8 and 834.1 are all due to oxygen; those whose wave-lengths he gives as 904.6, 977.2, 1010.6, 1037.0, 1175.9 and 1247.9 are due to carbon; those whose wave-lengths he gives as 916.7, 991.1 and 1085.5 are due to nitrogen; while those whose wave-lengths he gives as 972.7, 1026.0, 1216.0, are due to hydrogen.

c. The spectrum due to the 5 L-ring electrons of the nitrogen atom (atomic number 7) was obtained by using Al electrodes with ammonium nitrate in their cores and observing the new lines which were not due to aluminium, oxygen or hydrogen. Before this experiment was tried, it was predicted that the L_α line of nitrogen would have to lie between the L_α line of oxygen and that of carbon. It was found that the spectrum due to the nitrogen atom was very simple, beginning on our plates on the short wave-length side at 685.6 Å and *reaching a maximum in the line of wave-length 1085.3 Å, precisely as predicted.* This 1085.3 line is taken, in accordance with the foregoing convention, as the L_α line of the nitrogen atom. The only other strong lines which we have obtained which are due to this atom have the following wave-lengths: 685.6, 916.2, 991.1. The pair of nitrogen lines found by Lyman at 1492.8 and 1494.8 appear upon our plates, but they are very faint in comparison with the foregoing lines, the nitrogen spectrum thus showing a behavior quite like that due to the atoms of oxygen and carbon. Since the K_α line of nitrogen is at 31.2 Å, the ration of the K_α to the L_α frequency is 34.8.

d. In the case of *fluorine* a strong line has been found through the use of sodium fluoride at 657.2 Å. This is in about the position predicted by the foregoing mode of approach for its L_α line. Another strong fluorine line appears at 607.2 Å. These are the only lines thus far definitely identifiable as coming from the 7 L- ing electrons of the fluorine atom. The longer of these wave-lengths is taken as the L_α line of fluorine. Fluorine probably has, however, other lines of shorter wave-lengths but of such intensity that we have not yet been able to obtain them.

e. The spectrum, due to the 3 L-ring electrons of boron (atomic number 5) is especially interesting because of its simplicity. It contains less than 10 strong lines all told. It begins on the short wave-length side at 676.8 Å and has only the following strong lines: 676.8, 760.0, 1624.4, the two doublets at 2164.2, 2166.2 and 2496.9, 2497.8 and the single spark-line near the visible at 3451.5. According to the foregoing convention, its L_α line should be the doublet at 2497 Å. Since the K_α line of boron is at 67.2 Å, the ratio of the K_α to the L_α frequency in boron is about 37.

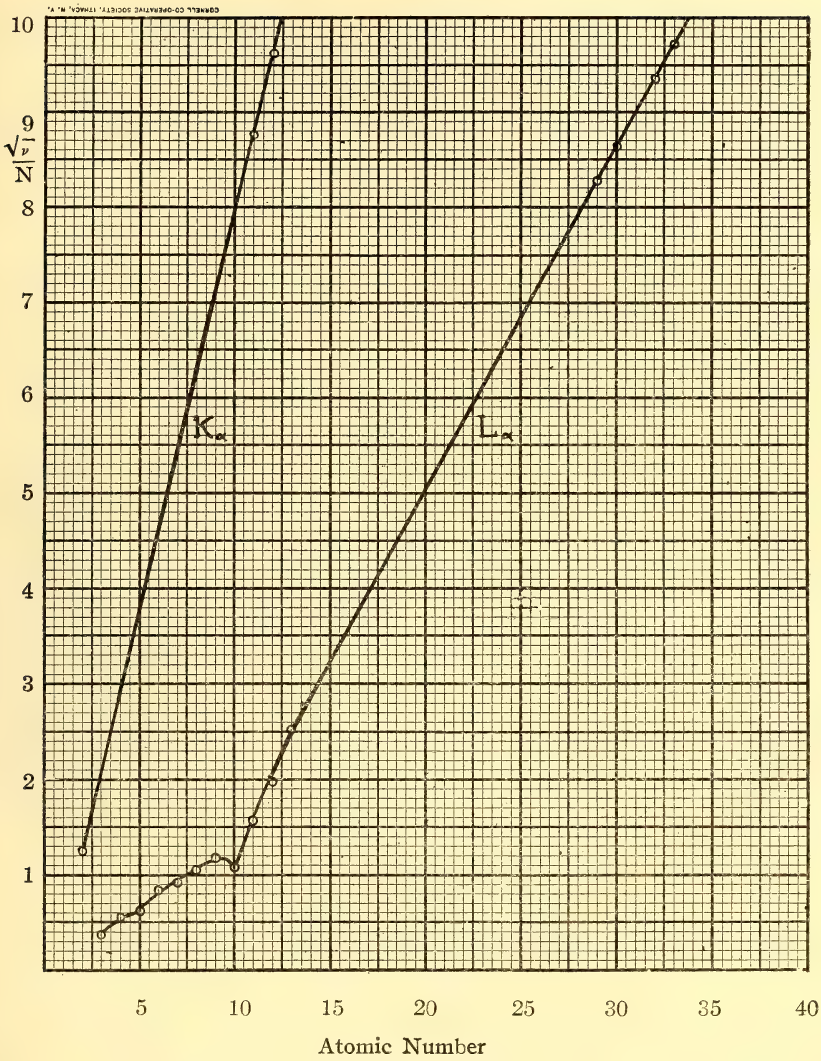
f. The spectrum due to the 2 L-ring electrons of *beryllium* begins on the short wave-length side, according to all the data available up to date, at 2175 Å and reaches its maximum, its L_α doublet, at 3130.6 and 3131.2. The entire absence upon our plates of any lines whatever due to beryllium between 230 Å and 2100 Å is a conspicuous illustration of the wide gaps in spectra obtained with ordinary gratings in or near the visible region.

g. Similiarly the present experiments with *lithium* (atomic number 3) reveal no lines whatever between the shortest wave-lengths measurable on the plates used in the case of lithium and the familiar series due to its single L-ring electron whose L_α line is at 6708 Å and whose convergence wave-length is at 2299 Å.

5. The graph of the L_α lines of the elements from lithium up is shown in the accompanying figure, along with the corresponding graph for the K_α lines from helium up taken from the work of others. *The only point on the L_α graph which has not been directly observed is that corresponding to neon which has been inserted from the resonance potential of that gas.*

It is to be observed that the method here employed gives, as I think, the characteristic spectra of the *atoms* of each element, not of the molecules. If it were possible to work with the *atomic gas* of each element, the L_α lines given herewith would be the *resonance* potentials of these atoms.

The progression thus revealed in these optical spectra is exceedingly interesting and simple, and very like that exhibited by X-ray spectra. The reason it has not been observed before is clearly because hitherto only the upper ends of these optical spectra have been observable, so that the unfolding of simple relationships between spectra and atomic number had to await the development of an ultra-violet technique.



L_α lines of Li, Be, B, C, N, O, F, Ne, Na, Mg and Al

A glance at the figure shows that we now have the complete outline of all of the types of radiations which are emitted by atoms of small atomic number. It remains only to fill in the details of their K_α and L_α spectra. One interesting fact which appears from a glance at the whole series of spectra of elements from hydrogen to neon is that atoms of odd ordinal all appear to have simple, few-lined spectra, while those of even ordinal number have much more complex spectra.

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[1] *Physic. Rev.*, **10,** 1917 (205).

[2] *Astroph. J.* **52,** 1920 (47).

[3] A detailed paper giving the full spectra of the light atoms with photographs will soon be published in the *Astrophysical Journal* in collaboration with Mr. I. S. Bowen.

[4] *Zeit. Physik,* **1,** 1920 (439).

[5] *Ibid.,* **2,** 1920 (470).

[6] This convention is more logical than that used in a former paper (cit.[2]) in naming the L_α line.

[7] *Astroph. J.,* **43,** 1916 (102).

THE AVERAGE OF AN ANALYTIC FUNCTIONAL AND THE BROWNIAN MOVEMENT

By Norbert Wiener

Department of Mathematics, Massachusetts Institute of Technology

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The simplest example of an average is the arithmetical mean. The arithmetical mean of a number of quantities is their sum, divided by their number. If a result is due to a number of causes whose contribution to the result is simply additive, then the result will remain unchanged if for each of these causes is substituted their mean.

Now, the causes contributing to an effect may be infinite in number and in this case the ordinary definition of the mean breaks down. In this case some sort of measure may be used to replace number, integration to replace summation, and the notion of mean reappears in a generalized form. For instance, the distance form one end of a rod to its center of gravity is the mean of its length with reference to its mass, and may be written in the form

$$\int l dm \div \int dm$$

where l stands for length and m for mass. It is to be noted that l is a function of m, and that the mean we are defining is the mean of a function. Furthermore, the quantity, here the mass, in terms of which the mean is taken, is a necessary part of its definition. We must assume, that is, a normal distribution of some quantity to begin with, in this case of mass.

The mean just discussed is not confined to functions of one variable; it admits of an obvious generalization to functions of several variables. Now, there is a very important generalization of the notion of a function of several variables: the function of a line. For example, the attraction of a charged wire on a unit charge in a given position depends on its shape. The length and area of a curve between two given ordinates depend on its shape. As a curve is essentially a function, these functions of lines may be regarded as functions of functions, and as such are known as functionals. Since a function is determined when its value is known for all arguments, a functional depends on an infinity of numerical determinations, and may hence be regarded as in some wise a function of infinitely many variables.

To determine the average value of a functional, then seems a reasonable problem, provided that we have some convention as to what constitutes a normal distribution of the functions that form its arguments. Two essentially different discussions have been given of this matter: one, by Gâteaux, being a direct generalization of the ordinary mean in n-space;[1] the other, by the author of this paper,[2] involving considerations from the theory of probabilities. The author assumes that the functions $f(t)$ that form the arguments of his functionals have as their arguments the time, and that in any interval of the small length ϵ as many receive increments of value as decrements of equal size. He also assumes that the likelihood that a particle receive a given increment or decrement is independent of its entire antecedent history.

When a particle is acted on by the Brownian movement, it is in a motion due to the impacts of the molecules of the fluid in which it is suspended. While the retardation a particle receives when moving in a fluid is of course due to the action of the individual particles of the fluid, it seems natural to treat the Brownian movement, in a first approximation, as an effect due to two distinguishable causes: (1) a series of impacts received by a particle, dependent only on the time during which the particle is exposed to collisions; (2) a damping effect, dependent on the velocity of the particle. If we consider one component of the total impulse received by a particle under heading (1), we see that it may be considered as a function of the time, and that it will have the sort of distribution which will make our theory of the average of a functional applicable.

It will result directly from the previous paper of the author that if $f(t)$ is the total impulse received by a particle in a given direction when the unit of time is so chosen that the probability that $f(t)$ lie between a and b is

$$\frac{1}{\sqrt{\pi t}}\int_a^b e^{-\frac{x^2}{t}}\,dx$$

then the average value of

$$A + \int_a^b f(t)\,G(t)dt + \int_a^b\int_a^b f(s)\,f(t)\,H(s,t)\,ds\,dt \quad [H(s,t) = H(t,s)]$$

will be

$$A + \int_a^b\int_a^s t\,H(s,t)\,ds\,dt. \tag{1}$$

We now proceed to a more precise and detailed treatment of the question.

2. Einstein[3] has given as the formula for the mean square displacement in a given direction of a spherical particle of radius r in a medium of viscosity η over a time t, under the action of the Brownian movement, the formula

$$\overline{d_t^2} = RTt \div 3\pi r\eta N$$

where R is the gas-constant, T the absolute temperature of the medium, and N the number of molecules per gram-molecule. In the deduction of this formula, Einstein makes two important assumptions. The first is that Stokes' law holds concerning forces of diffusion. Stokes' law states that a force F will carry particle of radius r through a fluid of viscosity η with velocity $F \div 6\pi r\eta$. Einstein's second assumption is that the displacement of a particle in some interval of time small in comparison with those which we can observe is independent, to all intents and purposes, of its entire antecedent history. It is the purpose of this paper to show that even without this assumption, under some very natural further hypotheses, the departure of $\overline{d_t^2}/t$ from constancy will be far too small to observe.

In this connection, it is well to take note of just what the Brownian movement is, and of the precise sense in which Stokes' law holds of particles undergoing a Brownian movement. In the study of the Brownian movement, our attention is first attracted by the enormous discrepancy between the apparent velocity of the particles and that which must animate them if, as seems probable, the mean kinetic energy of each particle is the same as that of a molecule of the gas. This discrepancy is of course due to the fact that the actual path of each particle is of the most extreme sinuosity, so that the observed velocity is almost in no relation to the true velocity. Now, Stokes' law is always applied with reference to movements at least as slow as the microscopically observable motions of a particle. It hence turns out that Stokes' law must be treated as a sort of average effect, or in the words of Perrin,[4] "When a force, constant in magnitude and direction, acts in a fluid on a granule agitated by the Brownian movement, the displacement of the granule, which is perfectly irregular at right angles to the force, takes in the direction of the force a component progressively increasing with the time and in the mean equal to $Ft \div 6\pi\zeta a$, F indicating the force, t the time, ζ the viscosity of the fluid, and a the radius of the granule."

It is a not unnatural interpretation of this statement to suppose that we may assume the validity of Stokes' law for the slower motions which are all that we see directly of the Brownian motion, so that we may regard the Brownian movement as made up (1) of a large number of very brief, independent impulses acting on each particle and (2) of a continual damping action on the resulting velocity in accordance with Stokes' law. It is to be noted that the processes which we treat as impulsive forces need not be the simple results of the collision of individual molecules with the particle, but may be highly complicated processes, involving intricate interactions between the particle and the surrounding molecules. It may readily be shown by a numerical computation that this is the case.

It follows from considerations discussed at the beginning of my paper on *The Average of an Analytic Functional* that after a time the probability that the total momentum acquired by a particle from the impacts of molecules will lie between x_0 and x_1 is of the form

$$\frac{1}{\sqrt{\pi ct}}\int_a^b e^{-\frac{x^2}{ct}}dx\cdot$$

Superimposed on this momentum is that due to the viscosity acting in accordance with Stokes' law, namely $6\pi r\eta V$, where V is the velocity of the particle. We shall write Q for $6\pi r\eta \div M$, where M is the mass of a particle.

Let us write τ for ct. Let the total impulse received by a given particle in time t, neglecting the action of viscosity, be $f(\tau)$. Consider $m(t)$, the actual momentum of the particle, as a function of t. Then

$$m(t+dt) = m(t) + f(ct+cdt) - f(ct) - Qm(ct + c\vartheta dt)dt \quad (o \leq \vartheta \leq 1).$$

We cannot treat this as a differential equation, as we have no reason to suppose that f has a derivative. We can make it into an integral equation, however, which will read

$$m(t) - m(o) = f(ct) - Q\int_o^t m(t)dt.$$

Clearly one solution of this integral equation is

$$m(t) = m(o)e^{-Qt} + f(ct) - Qe^{-Qt}\int_o^t f(ct)e^{Qt}dt,$$

and there is no difficulty in showing that an integral equation of this sort can have only one continuous solution.

Another integration gives for the distance traversed by the particle in time t

$$\begin{aligned} d_t &= \frac{1}{M}\left\{\left[-\frac{m(o)e^{-Qt}}{Q}\right]_o^t + \int_o^t f(ct)dt - \int_o^t Qe^{-Qt}\left[\int_o^t e^{Qt}f(ct)dt\right]dt\right\} \\ &= \frac{1}{M}\left\{\frac{m(o)}{Q}(1-e^{-Qt}) + e^{-Qt}\int_o^t e^{Qt}f(ct)dt\right\} \qquad (2) \\ &= \frac{1}{M}\left\{\frac{m(o)}{Q}(1-e^{-Qt}) + \frac{e^{-Qt}}{c}\int_o^\tau e^{\frac{Q\tau}{c}}f(\tau)d\tau\right\} \end{aligned}$$

Applying the methods of my previous paper, we get for the mean value of d_t^2, in accordance with (1);

$$\begin{aligned} \overline{d_t^2} &= \frac{1}{M^2}\left[\frac{m(o)}{Q}(1-e^{-Qt})\right]^2 + \frac{1}{M^2c^2}e^{-2Qt}\int_o^{ct}\int_o^{ct} A\left\{f(x)f(y)\right\}e^{\frac{Q(x+y)}{c}}dydx \\ &= \frac{1}{M^2}\left[\frac{m(o)}{Q}(1-e^{-Qt})\right]^2 + \frac{1}{M^2c^2}e^{-2Qt}\int_o^{ct}\left[\int_o^x ye^{\frac{Q(x+y)}{c}}dy\right]dx \\ &= \left[\frac{m(o)}{MQ}(1-e^{-Qt})\right]^2 + \frac{e^{-2Qt}}{M^2c^2}\int_o^{ct}\left[\frac{cx}{Q}e^{\frac{2Qx}{c}} - \frac{c^2}{Q^2}e^{\frac{2Qx}{c}} + \frac{c^2}{Q^2}e^{\frac{Qx}{c}}\right]dx \qquad (3) \\ &= \left[\frac{m(o)}{MQ}(1-e^{-Qt})\right]^2 + \frac{c}{4M^2Q^3}\left[2Qt - 3 + 4e^{-Qt} - e^{-2Qt}\right] \end{aligned}$$

Therefore

$$\left|\overline{d_t^2}/t-c/(2M^2Q^2)\right|\leq 1/t\left[m(o)(1-e^{-Qt})/(MQ)\right]^2$$

$$+\,c(3-e^{-Qt})(1-e^{-Qt})/(4M^2Q^3t)$$

$$\leq\left[m(o)/(MQ)\right]^2+3c/(4M^2Q^3)$$

This represents the absolute departure of $\overline{d_t^2}/t$ from constancy. Writing v for $m(o)/M$, the initial velocity of the particle, we get

$$\frac{|\overline{d_t^2}/t-c/(2M^2Q^2)|}{c/(2M^2Q^2)}\leq\frac{v^2/Q^2}{c/2M^2Q^2}+3/2Q$$

This is a measure of the relative departure of $\overline{d_t^2}/t$ from constancy. v cannot exceed, on the average, the velocity given on the average to the particle on the basis of the equipartition of energy; actually it is much smaller. $c/(2M^2Q^2)$ can be found directly, as it is nearly the observed value of $\overline{d_t^2}/t$. Q can be readily computed from the constants of the particles. Taking as a typical case one of Perrin's experiments on gamboge, Q turns out to be of the order of magnitude of 10^8, $c/2M^2Q^2$ of the order of magnitude of 10^{-8}, and the kinetic energy velocity of the order of magnitude of 10^{-1}. Hence the proportionate error is of the order of magnitude of 10^{-8}.

A proportionate error thus small is quite beyond the reach of our methods of measurement, so that we are compelled to conclude that $\overline{d_t^2}/t$, under the hypotheses we have here formulated, is sensibly constant. There are cases, however, which seem to give a slightly different value of $\overline{d_t^2}/t$ for small values of the time than for larger values. The explanation has been suggested[5] that over small periods the Einstein independence of an interval on previous intervals does not hold. The result of the present paper would be to suggest strongly, if not to demonstrate, that the source of the discrepancy, if, as appears, it is genuine, and not due to experimental error, is in the fact that Stokes' law itself is only a rough approximation, and that the resistance does not vary strictly as the velocity.

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[1] *Paris, Bull. Soc. Math. France,* 1919, pp. 47–70.

[2] "The Average of an Analytic Functional," in the last number of these PROCEEDINGS.

[3] *Leipzig, Ann. Physik,* **17,** 1905 (549).

[4] *Ann. Chim. Phys.*, Sept., 1909; tr. by F. Soddy.

[5] Cf. Kleeman, *A Kinetic Theory of Gases and Liquids,* §§ 56, 60.

MEASUREMENTS OF THE DEVIATION FROM OHM'S LAW IN METALS AT HIGH CURRENT DENSITIES

By P. W. Bridgman

Jefferson Physical Laboratory, Harvard University

Communicated August 10, 1921

Any picture of the mechanism of current conduction in metals which takes account of the part played by the electrons would lead to the expectation of departures from Ohm's law at high current densities. On the classical free electron basis J. J. Thomson has shown that at currents of the order of 10^9 amp./cm^2. the current would be expected to increase as the square root of the applied E. M. F., and hence that the resistance will increase indefinitely. Many attempts have been made to detect the existence of this effect experimentally, but without success. The chief source of difficulty has been the necessity for separating the change of resistance due to the great temperature rise under the heavy current from the change due to a departure from Ohm's law. The best known attempt in this direction is perhaps that of Maxwell.[1] Assuming that the departure from Ohm's law must be proportional to the square of the current, which is plausible on grounds of symmetry, he showed that at a density of 1 amp./cm^2. the resistance of platinum, iron, and German silver does not differ by more than 1 part in 10^{12} from the resistance at infinitely small currents. His maximum density was about 5×10^4 amp./cm^2.

By the application of a new method I have been able to eliminate the source of error due to temperature rise, to detect the existence of the effect, and to measure it with a fair degree of accuracy. The specimen is made one of the arms of a bridge, and is traversed simultaneously by a heavy direct current and a small superposed alternating current of acoustical frequency. The resistance of the specimen to the direct current is measured with an ordinary galvanometer, and the resistance to the alternating current is measured at the same time with a telephone. If there is a departure from Ohm's law under the heavy current, that is if the relation between current and E. M. F. is not linear, the two resistances will not be equal, and from their difference the departure from Ohm's law may be calculated. The reason for

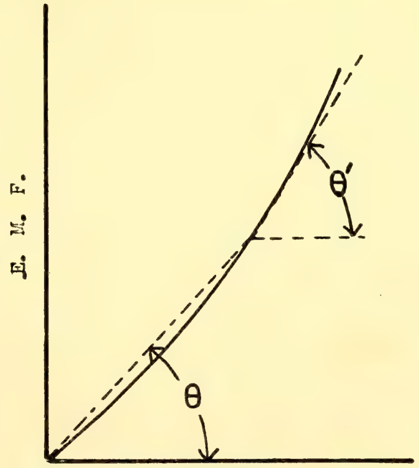

FIG. 1

this will be evident from an inspection of figure 1. It is to be remembered that a bridge is an instrument for balancing potentials in different parts of a net-work. At D. C. balance the potential drop corresponding to $\tan\theta$ in figure 1 is measured and at A. C. balance the potential drop corresponding to $\tan\theta'$.

It is evident that the direct effect of the unknown rise of temperature is eliminated by this method, because the temperature of the conductor is the same to both the direct and alternating current. There is, however, an indirect effect which is important, and which must be eliminated. Under the joint action of the direct and alternating current the wire receives a comparatively large supply of heat steadily, with a small heating and cooling effect superposed. Under this superposed heating and cooling the wire experiences alternations of temperature which give rise to fluctuations of resistance. The large direct current flowing through the fluctuating resistance gives rise to an alternating difference of potential in one of the bridge arms, which affects the A. C. balance. This action is like that of a microphone. It becomes vanishingly small at high frequencies of the alternating current, because the fluctuations of temperature become vanishingly small under these conditions.

The "microphone" effect was eliminated by making readings at a number of frequencies, plotting against the reciprocal of frequency and extrapolating to zero (that is, infinite frequency). The range of frequencies employed was from 320 to 3750 cycles per second. The extrapolation is therefore over a range only one-tenth of the observed readings. As further adding to the certainty of the extrapolation, it may be shown by a dimensional argument that the curve extrapolates to zero as a straight line. The extrapolated difference between D. C. and A. C. resistance gives the sought for departure from Ohm's law.

Measurements were made on gold and silver. These metals were in the form of thin leaf, cemented to a glass backing, cut into the shape of a narrow isthmus at the part intended to carry the high current density, and cooled by a stream of distilled water flowing over the glass. Two thicknesses of gold were used, 8×10^{-6} and 1.67×10^{-5}, and one thickness of silver, 2×10^{-5} cm. Greater thicknesses of gold were tried, but good results could not be obtained. It was possible to reach current densities up to about 5×10^{-6} amp./cm^2.

Further details of the experiment and of the electrical arrangements, which were sufficiently simple and obvious, will be described elsewhere, probably in the *Proc. Amer. Acad. Arts. and Sci.*

All of the experimental results obtained on eleven different samples of 8×10^{-6} gold (three of these were films formed by cathode deposit) are collected in figure 2. The ordinates are the extrapolated difference between D. C. and A C. resistance in terms of the initial resistance, and the abscissae are current densities in 10^6 amp./cm^2. The specimens varied in breadth from 0.06 to 0.22 mm. There was no correlation between the results and the breadth, although the microphone effect has a very strong connection. The results are scattering, but perhaps not more than would be expected when the magnitude of the current densities and the fact that different samples of gold leaf may differ in specific resistance by a factor of 2 or more is considered.

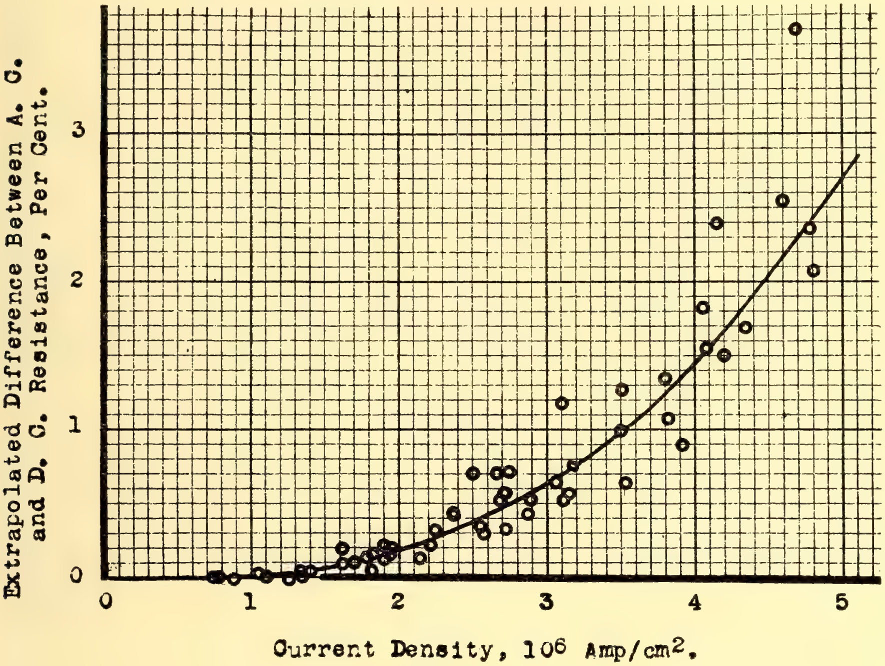

FIG. 2

The curve drawn through the observed points is taken as the best mean of the experimental results. Let us denote the equation of this curve by $\varphi(x)$. Then the departure from Ohm's law is given by the expression $\int \varphi(x)dx/x$. This will be proved in the detailed paper. This integral may be calculated graphically from the observed points.

In figure 3 is given the departure from Ohm's law, calculated in this way, for gold of two thicknesses and for silver. The departure is positive, that is, the resistance is greater at high density. For thin gold and silver the departure rises to something of the order of 1% at a density of 5×10^{-6} amp./cm^2; for the thicker gold it is greater. It was not possible to reach as high current densities in the thick as the thinner gold. The accuracy is greatest for thin and least for thick gold.

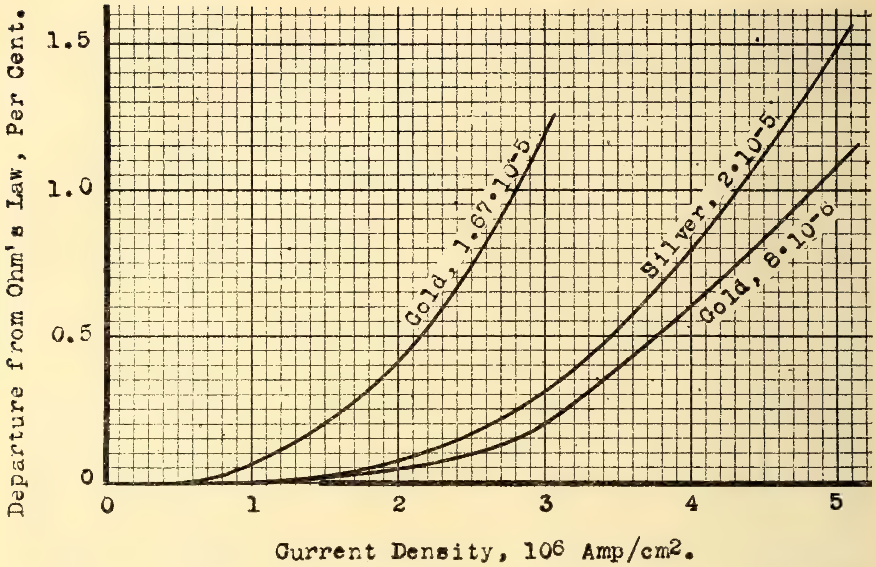

FIG. 3

The form of the curve is not that supposed as a first approximation by Maxwell, but the departure rises more rapidly than the square of the current. In fact the form of the curves seems to suggest infinitely high order of contact with axis at the origin. If however, for the purpose of numerical comparison we assume that below current densities of 10^6 amp./cm^2. the departure from Ohm's law is proportional to the square of the current within the limits of error, than the curve for thin gold shows that for it the resistance at 1 amp./cm^2. cannot differ by more than 1 part in 10^{16} from the resistance at infinitely small density (against 1 part in 10^{12} of Maxwell).

Theoretically the existence of this effect at currents of the order of 10^6 amp./cm^2. means that, granted a free path mechanism of conduction at all, the free path must be much longer than supposed on the classical basis, and accordingly the number of free electrons much less. Now on other grounds I have been led to the belief that conduction is by means of a free path mechanism,[2] of a different sort than that supposed in the classical theory, and that the number of electrons is much less and their paths longer than according to that theory. These new results are now in accord with this point of view. I have not been able as yet to carry through a more exact discussion on the free path basis and to obtain the numerical value for the length of the free path which these results would involve.

The difference in results for gold of different thickness is also what would be expected. The results for thick gold are not nearly as accurate as for the thinner, but there can seem no question but that the departure is really greater for thick than for thin gold. If the free path is of the order of 10^{-5} cm., that is, comparable with the thickness of the leaf, then the average path in the thicker metal will be longer than in the thinner, and a greater departure would be expected in the thicker metal, as found.

[I am much indebted to my assistant, Mr. J. C. Slater, for his skill in making the readings.]

[1] Maxwell, C., Everett, J. D. and Schuster, A., *B. A. Rep.* 1876 (36–63).

[2] Bridgman, P. W., *Physic. Rev., Ithaca,* (2), **17,** 1921 (161–194).

AN INTEGRAL EQUALITY AND ITS APPLICATIONS

BY EINAR HILLE

HARVARD UNIVERSITY

Communicated by G. D. Birkhoff, May 15, 1921

The purpose of this note[1] is to deduce an integral equality adjoined to a linear homogeneous differential equation of the second order and to show some applications of such equalities to the question of the distribution of zeros of solutions of such differential equations in the complex plane.

Let $G(z)$ and $K(z)$ be two single-valued and analytic functions of z throughout the region under consideration and take the differential equation

$$\frac{d}{dz}\left[K(z)\frac{dw}{dz}\right] + G(z)w = 0, \tag{1}$$

or the equivalent system

$$\left\{\begin{aligned} \frac{dw_1}{dz} &= \frac{w_2}{K(z)}, \\ \frac{dw_2}{dz} &= -G(z)w_1, \end{aligned}\right. \qquad \text{with} \qquad \left\{\begin{aligned} w_1 &= w, \\ w_2 &= K(z)\frac{dw}{dz}. \end{aligned}\right. \tag{2}$$

From this system we easily deduce the relation

$$\left[\bar{w}_1 w_2\right]_{z_1}^{z_2} - \int_{z_1}^{z_2} |w_2|^2 \overline{\left[\frac{dz}{K(z)}\right]} + \int_{z_1}^{z_2} |w_1|^2 G(z)dz = 0, \tag{3}$$

where $\bar{u}$ denotes the conjugate of u. If we put

$$d\mathrm{K} = \frac{dz}{K(z)}, \quad d\Gamma = G(z)dz, \tag{4}$$

relation (3) becomes

$$\left[\bar{w}_1 w_2\right]_{z_1}^{z_2} - \int_{z_1}^{z_2} |w_2|^2\, \overline{d\mathrm{K}} + \int_{z_1}^{z_2} |w_1|^2 d\Gamma = 0, \tag{5}$$

or, split up into real and imaginary parts,

$$(6) \qquad \Re\left[\overline{w_1}w_2\right]_{z_1}^{z_2} - \int_{z_1}^{z_2} |w_2|^2\, dK_1 + \int_{z_1}^{z_2} |w_1|^2\, d\Gamma_1 = 0,$$

$$(7) \qquad \Im\left[\overline{w_1}w_2\right]_{z_1}^{z_2} + \int_{z_1}^{z_2} |w_2|^2\, dK_2 + \int_{z_1}^{z_2} |w_1|^2\, d\Gamma_2 = 0,$$

where

$$(8) \qquad dK = dK_1 + idK_2;\ d\Gamma = d\Gamma_1 + id\,\Gamma_2.$$

We call relation (5) *Green's transform* of the differential equation (1). This relation can be used in many ways for obtaining information concerning the distribution of the zeros of a function $w(z)$, satisfying (1). Formula (5) enables us to assign regions, below called *zero-free domains* where there can be no zeros of $w(z)$ or $\frac{dw}{dz}$. Some of these ways are indicated below.

The four differential equations

$$(9) \qquad dK_1 = 0,\ dK_2 = 0;\ d\Gamma_1 = 0,\ d\Gamma_2 = 0,$$

define four families of curves; the K_1-family and the K_2-family forming the K-net and the Γ_1 and Γ_2-families forming the Γ-net. The two families belonging to the same base net are orthogonal trajectories of each other.

Take a solution $w_a(z)$ of (1) such that

$$W_a(z) = K(z)\, w_a(z) \frac{dw_a}{dz}$$

vanishes at a regular point a in the complex plane. Construct the Riemann surface on which $w_a(z)$ is single-valued and mark the two base nets on the surface. Further, draw all curves on the surface, starting from $z = a$ which do not pass through any of the singular points of the differential equation and which are composed of arcs of curves of the two base nets such that along the whole path one and the same of the following four inequalities is fulfilled, namely

$$(10) \quad 1^\circ \left\{ \begin{array}{l} dK_1 \geq 0, \\ d\Gamma_1 \leq 0; \end{array} \right| \quad 2^\circ \left\{ \begin{array}{l} dK_1 \leq 0, \\ d\Gamma_1 \geq 0; \end{array} \right| \quad 3^\circ \left\{ \begin{array}{l} dK_2 \geq 0, \\ d\Gamma_2 \geq 0; \end{array} \right| \quad 4^\circ \left\{ \begin{array}{l} dK_2 \leq 0, \\ d\Gamma_2 \leq 0. \end{array} \right.$$

We agree to smooth the corners of the path-curves, when necessary, in order to preserve the continuity of the tangent along the curve. Such curves we call *standard paths* and designate the four different kinds (distinguished by their characteristic inequalities) respectively

$$(11) \qquad SK_1^+,\ SK_1^-,\ SK_2^+ \text{ and } SK_2^-.$$

The points on the surface which belong to at least one standard path, emanating from $z = a$, together form the *standard domain* $D(a)$ of a. A point on the boundary of $D(a)$ is counted part of the standard domain provided it is not a singular point of the differential equation and is different from a itself. In view of (6) and (7) we obtain the result:

There is no zero of $W_a(z)$ *in the standard domain of* a.

Here is another application of the Green's transform. Suppose $G(z)$ and $K(z)$ to be analytic and, furthermore, real on an interval (a,b) of the real axis. Then there are solutions of (1) which are real on the same interval. Draw all standard paths of the third and the fourth kind, SK_2^+ and SK_2^-, which emanate from the points on (a,b). The points on these paths, not including possible singular points and the points on (a,b), form a zero-free region for all solutions real on (a,b) in virtue of formula (7). For this statement we have only used the fact that for real solutions $\mathfrak{Im}\,[\overline{w_1(z)}\; w_2\,(z)] = 0$ on (a,b). The generalizations are obvious.

The Standard Domain Is Covariant under Conformal Transformation.—By this we understand that a change of independent variable from z to Z by putting $Z = F\,(z)$ which preserves the form of the differential system (2), carries the standard domain of a point z_0 in the z-plane over into the standard domain of the corresponding point Z_0 in the Z-plane and these two domains are the conformal maps of each other by the transformation $Z = F\,(z)$ and its inverse, provided $F\,(z)$ is regular in $D\,(z_0)$ and $F'\,(z) \neq 0$ there. This is obvious because Green's transform is an invariant under such a transformation and the base nets in the two planes correspond to each other by the conformal transformation.

The standard domain is not covariant under a simultaneous change of dependent and independent variable. By a fortunate choice of such variables it is often possible to determine zero-free domains of such an extent that there is comparatively little freedom left for placing possible zeros; thus one is able to obtain a fairly good qualitative description of the arrangement of the zeros in the plane. The transformation

$$(12)\quad \begin{cases} dZ = \sqrt{\dfrac{G(z)}{K(z)}}\, dz, \\ W = \sqrt{S(Z)}\, w, \\ S(Z) = \sqrt{G(z)K(z)}, \end{cases}$$

often yields good [illegible]e for investigation of the distribution of zeros of solutions in the neighborhood of an irregular singular point of the differential equation.

[1] This note is an abstract of a paper offered to *Trans. Amer. Math. Soc.* for publication.

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PROCEEDINGS
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THE MOBILITIES OF ELECTRONS

By Leonard B. Loeb*

Ryerson Physical Laboratory, University of Chicago

Communicated by R. A. Millikan, Aug. 17, 1921

In a recent paper[1] it was shown that the electrons could presumably travel through N_2 and H_2 gases in a free state at atmospheric pressure. It was found in some cases impossible to evaluate the mobilities of these electrons with the low frequencies of commutation available. The values of the mobilities indicated by these results appeared to be so much higher than the values obtained by earlier investigators that it was felt worth while to attempt to measure them accurately.

The mobilities of electrons were accordingly determined in N_2 using the high frequency oscillations obtained from two Western Electric Company "E" tube oscillators operating in parallel. The diagram of connections is shown in figure 1. In some cases the oscillations were taken directly from the primary condenser C_1, and at other times from the condenser C_2 of a secondary circuit tuned to resonance with the primary circuit. The method of measurement was the well known Rutherford alternating current method. The electrons used where photo electrons liberated by ultraviolet light from one of the plates. The ionization chamber used was similar to the one used in measurements of the constant of attachment of electrons to gas molecules,[2] except that still greater precautions were taken in this chamber to avoid contamination. The measurements were made in the same manner as were the measurements for the determination of the coefficient of attachment. The current to the electrometer plate was measured as a function of the value of the alternating potential between the plates. This current was then plotted against the potential difference thus yielding a mobility curve. The potential difference V_0, at which this curve cut the axis was substituted in the equation,

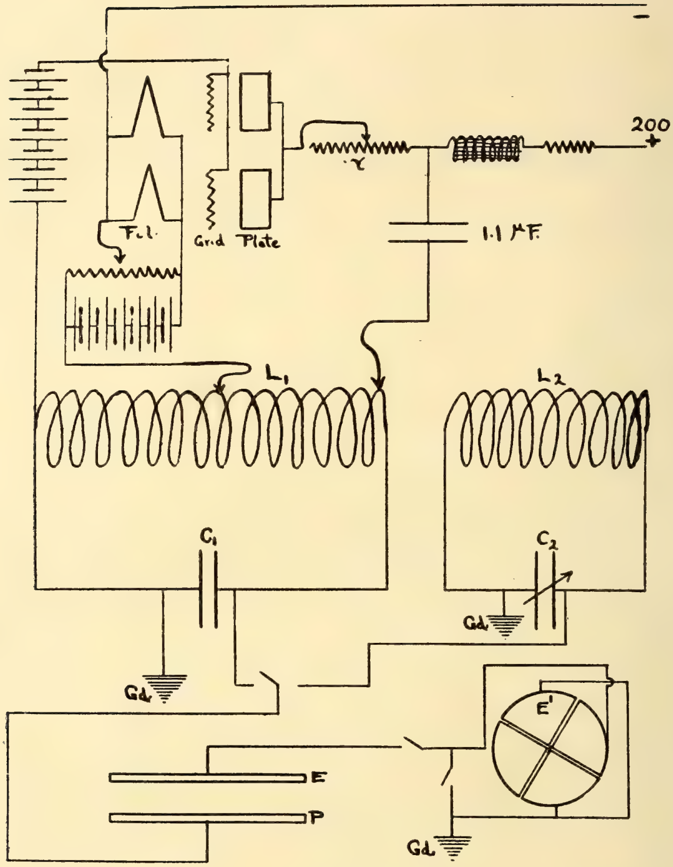

FIG. 1

$$U = \frac{\pi n d^2}{\sqrt{2V_0}},$$

in order to evaluate the mobility U. In this equation n is the frequency of the alternations, d is the distance between the plates in cm. and V_0 is the voltage in volts as read from the static voltmeter. In the oscillating circuits the potential was varied by varying the resistance r of the plate circuit. Since varying r altered the frequency n slightly the frequency of the oscillations was determined by means of a wave meter for several values of the voltage used in each determination. The experiments reported here covered a range of pressures from 600 mm. to 75 mm. The frequencies used varied from 7000 cycles per second to 150,000 cycles. The voltages used were varied from 10 volts to 300 volts and the plate distances lay between 2.0 cm. and 1.5 cm. The values of U thus obtained were reduced to give the mobility constant K at atmospheric pressure by means of the equation, $K = (pU)/760$, where p is the pressure in mm. of mercury.

The mobilities found in this manner were astonishingly high. At 600 mm. mobilities of the order of magnitude of 15,000 cm. / sec. were obtained, while the highest previous mobility determined[3] in N_2 was 500 cm./sec. Furthermore the value of K obtained was not a constant as one would expect. K was found to be a function of the field strength and pressure. For a given pressure the value of K plotted as a function of V_0/d the field strength was found to lie on a hyperbola of the form, $K = \frac{a}{b + V/d}$. For different pressures a family of such hyperbolae were obtained which were expressed by the equation,

$$K = \frac{571{,}000}{21 + 760\, V_0/(pd)}.$$

This is shown in figure 2. The full curves are drawn from the above equation at values of p and d at which the measurements were made. The

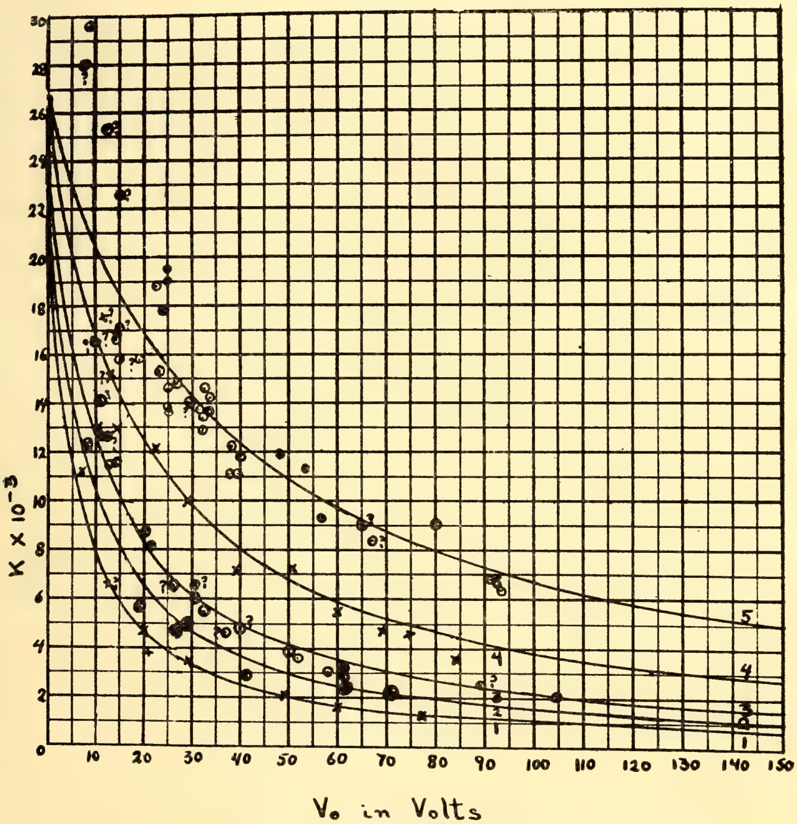

FIG. 2

points plotted are the actual experimental values of K obtained under the conditions of p, V_0, and d given. When the inaccuracies of such measurements are considered it is seen that the above equation represents the behavior of K to a satisfactory degree of approximation.

The success in measuring mobilities of such high values is probably due to the fact that especial care was taken to avoid contaminating gases, and to the fact that the frequencies employed were very high. The latter factor made it possible to measure the mobilities of purely electronic carriers only, for with the short intervals of time used only electrons which had made no attachments at all could succeed in crossing the plates. It seems likely that the low electronic mobilities observed by the previous workers were found for electrons which had been completely free for only a portion of their path between the plates because of the low frequencies of alternation used. The magnitude of the values obtained in these experiments is more nearly in accord with the values of electron mobilities predicted on the basis of the equations of Townsend[4] and Lenard[5] (i.e. of 6940 cm./sec. and 4260 cm./sec. respectively), than the earlier values.

The fact that K is not a constant is most interesting. It indicates that the term mobility constant has no significance for electrons; since their velocity in the field is no longer directly proportional to the field strength and inversely proportional to the pressure. The way in which K varies with V_0/d and with p indicates that the velocity of drift of the electrons in the direction of the field is influenced by the energy gained by the electron in the electrical field between impacts. A variation of the energy of the electron in the field such as would cause the observed variation of K can only occur when the electrons make *partially elastic* impacts with the gas molecules.

A more detailed account of these experiments will later appear elsewhere. The experiments are being extended to hydrogen and if possible to other gases.

---

* National Research Fellow of the NATIONAL RESEARCH COUNCIL

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A METHOD FOR OBTAINING CONSTANTS FOR FORMULAS OF ORGANIC GROWTH*

BY H. S. REED

UNIVERSITY OF CALIFORNIA

Communicated by R. Pearl, October 8, 1921

The growth of many organisms may be computed from the equation

$$\log \frac{x}{A - x} = K(t - t_1),$$

in which x represents the size of the organism at time t, A represents the size of the organism at maturity or at the end of a particular cycle of growth, t_1 is the time at which $x = A/2$, and K is a constant. This is the equation for an autocatalytic reaction and we are indebted to Robertson (1908) for showing its applicability to growth processes.

While the equation expresses the growth of plants with a high degree of accuracy, it often fails to fit the observed data in the early life of the organism. In other words, the computed values of x are too large when t is small. Recently this feature of the equation has been discussed by Mitscherlich (1919) and Rippel (1919).

The present paper intends to show how a simple graphic method may be used to overcome the difficulty just stated.

Let us assume that the true state of affairs is represented by the equation

$$\log \frac{x}{A - x} = K(t - t_1)^c.$$

This is merely assuming that the exponent of $(t - t_1)$ may or may not be unity. We may write

$$\log\left(\log \frac{x}{A-x}\right) = \log K + c \log (t - t_1)$$

This is the equation of a straight line if $\log\left(\log \frac{x}{A-x}\right)$ be used as ordinate and $\log (t - t_1)$ as abscissa. The intercept on the y-axis will be $\log K$ and the slope of the line c.

When $t < t_1$ the values of $(t - t_1)$ are negative in sign, as t approaches the value of t_1 the value of $(t - t_1)$ decreases to zero. As t increases beyond t_1 the values of $(t - t_1)$ are positive. However the sign of the quantity $(t - t_1)$ does not affect the logarithm. The result is that we actually get two lines on the chart, one for values of the equation when $t < t_1$ and another for values when $t > t_1$. These lines as determined from the observations, are seldom superimposed. With them as guides the "best" straight line is drawn from which the values of the slope and intercept are read off. If the lines make much of an angle with each other it usually means that the value of t_1 has been poorly chosen. The choice of a value for t_1 is facilitated if one ascertains all the values of $\log\left(\log \frac{x}{A-x}\right)$, and from them determines by inspection the values of $(t-t_1)$ which will bring approximately equivalent values of $\log\left(\log \frac{x}{A-x}\right)$ as nearly as possible the same distance from the ordinate. For this purpose, the highest and lowest values of $\log\left(\log \frac{x}{A-x}\right)$ are less reliable than the intermediate values.

A few examples may be given.

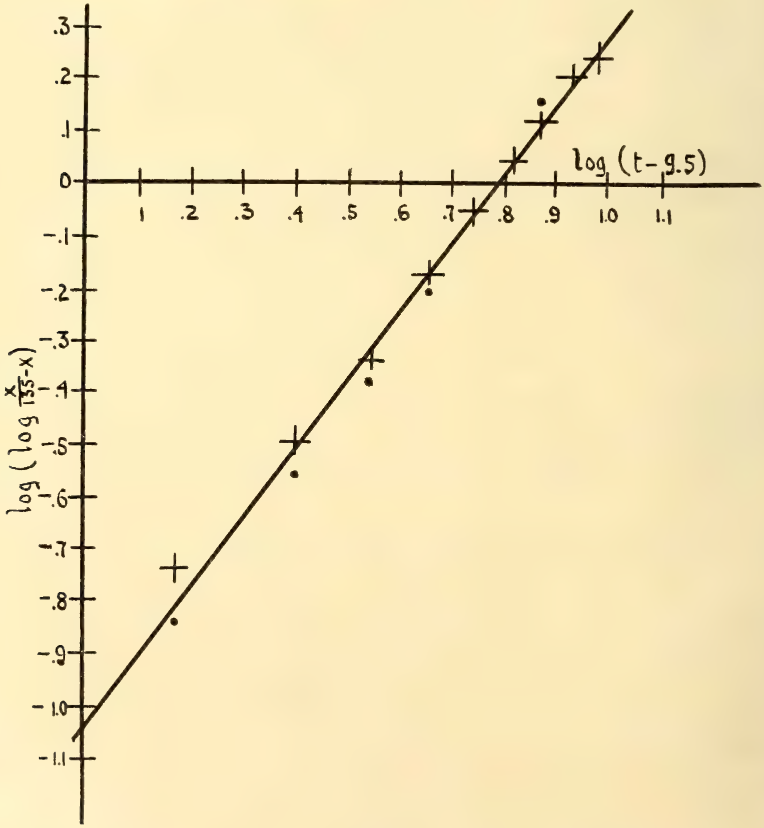

FIG. 1

Graph of values for lemon shoots. Points represent values of log $(t-9.5)$ when $t < t_1$; crosses represent same when $t > t_1$.

The average length of 37 shoots on a group of lemon trees is given in the column headed x in table 1. The measurements were made each week for the first 21 weeks of their growth. From the observed length at intervals of a week the values of

$$\log\left(\log\frac{x}{135-x}\right) = \log K + c\log(t - 9.5)$$

were ascertained and plotted (fig. 1). From the line thus obtained it was ascertained that $\log K = -1.05$ and $K = .0891$. The value of c as determined by the slope was 1.35.

Table 1 contains in the column headed x_1 the values computed from the equation

$$\log\frac{x}{135 - x} = .0891(t - 9.5)^{1.35}.$$

TABLE 1

GROWTH OF LEMON SHOOTS. COMPARISON OF OBSERVED AND CALCULATED VALUES

t WEEKS	x CM.	x_1 CM.	θ_1 CM.	x_2 CM.	θ_2 CM.
1	3.0	3.2	.2	5.3	2.3
2	4.9	5.7	.8	7.6	2.7
3	9.6	9.6	.0	10.8	1.2
4	15.6	12.8	−2.8	15.1	−0.5
5	25.8	23.4	−2.4	20.9	−4.9
6	37.3	33.4	−3.9	28.5	−8.8
7	46.8	44.8	−2.0	37.8	−9.0
8	56.5	55.6	−0.9	48.9	−7.6
9	65.2	61.4	−3.8	61.2	−4.0
10	73.5	73.6	.1	73.8	0.3
11	81.6	79.4	−2.2	86.1	4.5
12	91.4	90.2	−1.2	97.2	5.8
13	100.0	101.7	−1.7	106.5	6.5
14	110.9	111.6	.7	114.1	3.2
15	119.4	122.2	2.8	119.9	0.5
16	125.3	125.4	.1	124.2	−1.1
17	128.8	129.3	.5	127.4	−1.4
18	131.6	131.8	.2	129.7	−1.9
19	132.5	133.1	.6	131.4	−1.1
21	134.3	134.5	.2	133.2	−1.1
Root-mean-square deviation			1.88		4.40

The root-mean-square deviation of calculated from observed values was 1.88 cm.

In the column headed x_2 will be found values computed from the equation

$$\log\frac{x}{135-x} = .1635(t - 9.5).$$

The root-mean-square deviation of these values is 4.40 cm.

TABLE 2

GROWTH OF SHOOTS ON YOUNG PEAR TREES

t DAYS	x CM.	x_1 CM.	θ_1 CM.	x_2 CM.	θ_2 CM.
0		6.8	6.8	8.1	8.1
7	5	10.0	5.0	10.7	5.7
14	11	14.6	3.6	15.4	4.4
21	19	20.7	1.7	21.3	2.3
28	29	28.6	−0.4	28.8	−0.2
35	40	39.2	−0.8	38.1	−1.9
42	48	49.8	1.8	48.5	0.5
49	59	59.0	0.	54.4	−4.6
55	69	67.6	−1.4	69.0	0.
63	76	79.1	3.1	80.4	4.4
70	86	88.2	2.2	88.8	2.8
77	94	95.5	1.5	95.6	1.6
84	100	101.5	1.5	100.9	0.9
91	102	105.5	3.5	104.8	2.8
98	108	108.5	0.5	107.6	−0.4
105	110	110.5	0.5	109.6	−0.4
112	111	111.8	0.8	110.9	−0.1
119	112	112.5	0.5	111.9	−0.1
126	113	113.0	0.	112.6	−0.4
Root-mean-square deviation			2.38		3.14

In a recent paper (Reed, 1920) I showed that the growth of shoots on Bartlett pear trees followed the course of an autocatalytic reaction very closely, except in the first few weeks of the season. During the early part of the growth period the calculated values were larger than those observed. Using the observed data I have computed values of K and c from the equation

$$\log\left(\log\frac{x}{114 - x}\right) = \log K + c \log (t - 47.4).$$

Plotting the values of this equation (fig. 2) the value of log K (the intercept) is -1.75 ($K = .0178$) and c (the slope) is 1.09.

When the size of the pear shoots is computed from the equation

$$\log\frac{x}{114 - x} = .0178(t - 47.4)^{1.09}.$$

we obtain values agreeing more closely with the observed, as shown by table 2, where x = observed mean length of the young shoots at successive intervals; x_1 = length calculated from

$$\log\frac{x}{114 - x} = .0178(t - 47.4)^{1.09}.$$

θ_1 = deviations of x_1 from x; x_2 = length calculated from

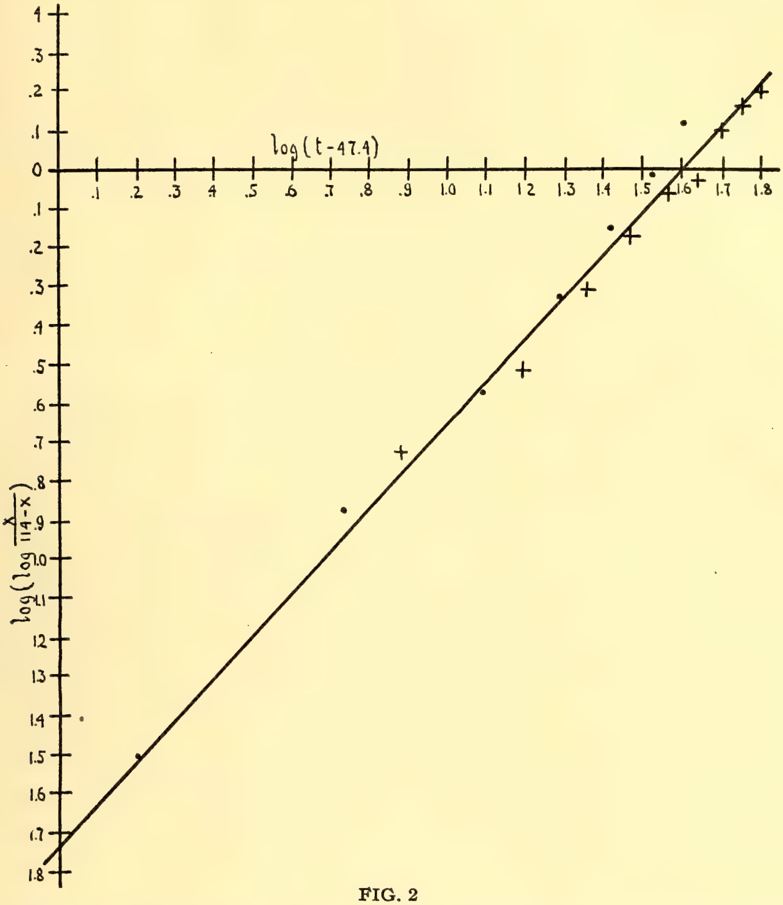

FIG. 2

Graph of values for pear shoots. Points represent values of log $(t - 47.4)$ when $t < t_1$; crosses represent same when $t > t_1$.

$$\log \frac{x}{114 - x} = .0242(t - 47.4)$$

θ_2 = deviations of x_2 from x. The root-mean-square deviation of x_1 from x is 2.38 cm., and that of x_2 from x is 3.14.

Finally, a growth process which closely coincided with values computed from the usual formula, may be mentioned. In the paper cited (Reed, 1920) the average height of a population of young walnut trees (*J. regia*) was presented. The calculated values had a root-mean-square deviation from the observed of only 3.12 cm. The equation for growth as derived by graphic method was

$$\log \frac{x}{153 - x} = .0208(t - 96)^{1.03}.$$

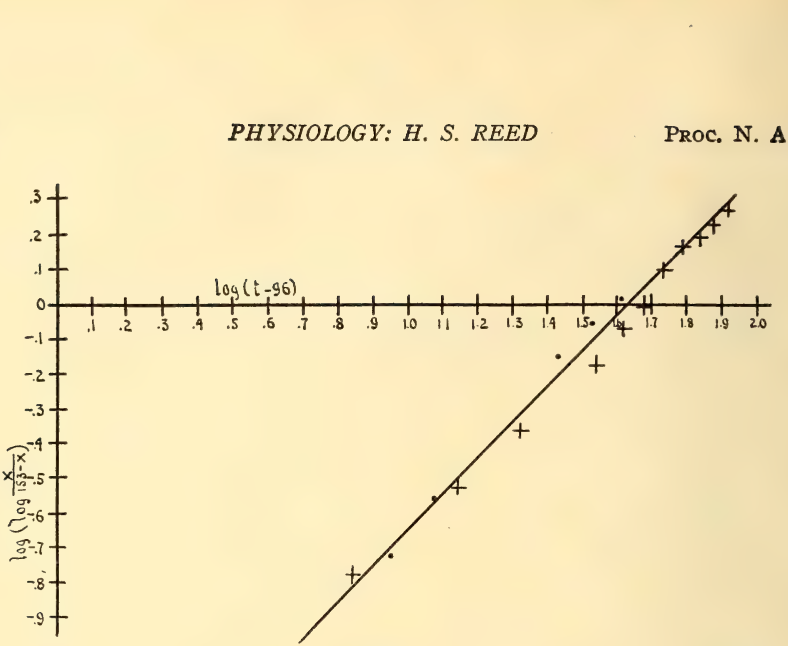

FIG. 3

Graph of values for walnut trees. Points represent values of log $(t - 96)$ when $t < t_1$; crosses represent same when $t > t_1$.

The value of K here is only .002 less than that first used and the exponent of $(t - 96)$ is practically unity. The root-mean-square deviation of values obtained from this equation was 2.65 cm., which differs so slightly from the first that there is no real advantage in favor of one equation over the other.

The use of the method described does, however, give a more accurate means of computing the growth in cases where the initial growth of the organisms falls measurably below that given by the equation as previously used.

* Paper No. 85, University of California, Graduate School of Tropical Agriculture and Citrus Experiment Station, Riverside, California.

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ON THE APPROXIMATE SOLUTIONS IN INTEGERS OF A SET OF LINEAR EQUATIONS

By H. F. Blichfeldt

Department of Mathematics, Stanford University, California

Read before the Academy, April 26, 1921

1. It has been proved that a set of $n+1$ integers[1] $x_1, x_2, \ldots, x_n, w$ exist which satisfy approximately the n equations[2]

$$x_1 - \alpha_1 w - \beta_1 = 0, \ldots, x_n - \alpha_n w - \beta_n = 0, \qquad (1)$$

where $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ are $2n$ given real numbers, the first n of which are subject to a condition (C) stated below. The degree of approximation is as follows: having specified a positive number ϵ as small as we please, then the numerical values of the left-hand members of the above equations (i.e., the "errors") can simultaneously be made smaller than ϵ. The condition (C) imposed upon $\alpha_1, \alpha_2, \ldots, \alpha_n$ is this: they are to be irrational numbers of such a nature that a linear expression of the form $a_0 + a_1\alpha_1 + \ldots + a_n\alpha_n$, with integral coefficients $a_0, a_1, \ldots, a_n$, can vanish exactly *only* when these coefficients are all $= 0$.

For a given approximate solution of the equations (1) the errors shall be designated, respectively, by $\epsilon_1, \ldots, \epsilon_n$.

2. In the two special cases: (I) $n = 1$, or (II) $\beta_1 = \beta_2 = \ldots = \beta_n = 0$, we can make additional demands upon the errors, namely (and this is of importance in applications) we can cause them to be smaller than certain corresponding functions of w. Thus, in the case (I), we can demand that $\epsilon_1 < \epsilon$ and at the same time that $\epsilon_1 < 1/(4\,|w|)$; in case (II) we can demand that $\epsilon_1, \ldots, \epsilon_n$ be each $< \epsilon$ and at the same time $< k/(|w|)^m$, where $k = n/(n+1)$, and $m = -(n-1)/n$; furthermore, the condition (C) is not necessary in case (II).[3] These results may be stated in another way: in case (I), α being irrational, an infinite number of sets of integers $(x', w'), (x'', w''), \ldots.$ exist which, when substituted in turn in $x - \alpha w - \beta = 0$, will produce errors $\epsilon', \epsilon'', \ldots.$ of such magnitudes that $\epsilon' < 1/(4\,|w'|)$, $\epsilon'' < 1/(4\,|w''|)$, etc. Similarly, in case (II) we have an infinite number of solutions satisfying corresponding inequalities.

3. In the general case the errors cannot be made to satisfy such extra demands. Stated more precisely: *no matter what function $f(w)$ be assumed, if only it be subjected to the conditions* (D) *below, we can always construct a set of equations* (1) *with attendant condition* (C), *which do not admit an infinite number of approximate solutions in integers* $(x_1, x_2, \ldots, x_n), \ldots.,$ *if we demand the following degree of accuracy for each solution:* $\epsilon_1 < f(|w|), \ldots, \epsilon_n < f(|w|)$. *In fact, we may even substitute the weaker demand.*

$$\epsilon_2\epsilon_3\ldots\epsilon_n + \epsilon_1\epsilon_3\ldots\epsilon_n + \ldots + \epsilon_1\epsilon_2\ldots\epsilon_{n-1} < f(\,|\,w\,|\,). \qquad (2)$$

The conditions (D) referred to are: for every positive integer w, $f(w)$ is a positive number which decreases when w increases, and approaches 0 as a limit when w tends to ∞.

4. We proceed to give an outline of the proof, limiting ourselves to the case $n = 3$, which is sufficiently illustrative. The function $f(w)$ being given subject to (D), we shall take as our equations the following set (under an obvious change of notation):

$$x - \alpha w - r\alpha = 0,\ y - \beta w - s\beta = 0,\ z - \gamma w - t\gamma = 0. \qquad (3)$$

Here r, s, t represent three arbitrary, but different integers, while α, β, γ are defined as the limits approached when j tends to ∞ by three series of generalized (Jacobian) continued fractions x_j/w_j, y_j/w_j, z_j/w_j; $j = 1, 2, \ldots.$ We take at the outset any four sets of non-negative integers $(x_1, \ldots, w_1)$, $\ldots$, $(x_4, \ldots, w_4)$, such that the determinant $(x_1y_2z_3w_4) = 1$; succeeding sets $(x_j, \ldots, w_j;\ j > 4)$ are constructed from previous sets by the rule: $x_j = s_{j-1}x_{j-1} + x_{j-4}$, $y_j = s_{j-1}y_{j-1} + y_{j-4}$, etc., involving a series of positive integers s_4, s_5, s_6, $\ldots,$ which, in turn, are expressed in terms of another series designated by [3], [4], [5], $\ldots$; namely, we put $s_j = 2^{[j]-[j-1]}$. The members of the new series are any set of positive integers satisfying the following conditions:

$[3] = 1$; when $j > 3$ take $[j] \geqq 3[j-1]$, $f(2^{[j]-[j-1]}) < 2^{-6[j-1]}$.

5. For convenience in the further development we make use of the letters M, m, μ to represent certain non-negative coefficients (constants or variables), the actual values of which are not required: M for a number having a lower bound > 0, m for one having an upper bound, and μ for one having both a lower bound > 0 and an upper bound.

We note the following preliminary relations:

$x_j = \mu.2^{[j-1]}$, $y_j = \mu.2^{[j-1]}$, etc.;

$\xi_j \equiv x_j - \alpha w_j = \pm m/s_j$, $\eta_j \equiv y_j - \beta w_j = \pm m/s_j$, $\zeta_j = z_j - \gamma w_j = \pm m/s_j$; $x_iw_j - x_jw_i = (x_iw_j) = \pm m.2^{[j-1]-[i]+[i-1]}$ when $j > i$, $y_iw_j - y_jw_i = (y_iw_j) =$ etc., etc. Moreover, (x_iw_j), etc., do not vanish when j is taken sufficiently high.

(The condition (C), §1, is satisfied by α, β, γ. For, an equation $a_0 + a_1\alpha + a_2\beta + a_3\gamma = 0$ would imply $a_0w_j + a_1x_j + a_2y_j + a_3z_j = 0$ for every subscript j above a certain number. But this requires $a_0 = a_1 = a_2 = a_3 = 0$, since the determinant $(x_jy_{j+1}z_{j+2}w_{j+3}) = \pm 1$.)

6. If x, y, z, w represent any four integers, then at least two of the numbers

$A = xw_j - x_j(w+r)$, $B = yw_j - y_j(w+s)$, $C = zw_j - z_j(w+t)$ are of magnitude $\pm M.2^{[j-1]-2[j-2]}$, when j exceeds a certain number. For selecting any pair, say A, B, we have

$$(Aq - Bp)\,w_{j-1} + (r - s)pq \equiv 0 \qquad (\text{mod. } w_j),$$

where $p=(x_jw_{j-1})$, $q=(y_jw_{j-1})$. But this congruence is found to imply $|Aq - Bp|w_{j-1} > wj$ for sufficiently high values of j. Hence, either $|Aq|$ or $|Bp|$ or both are $> w_j/w_{j-1}$.

7. Reverting now to the equations (3), we take any approximate solution in integers, say x, y, z, w. Assuming for our purpose that $|w| \geqq 2^{[4]-[3]}$, we determine the positive integer j such that

$$2^{[j-1]-[j-2]} \leqq |w| < 2^{[j]-[j-1]}.$$

We have

$$w_j\epsilon_1 = |A+(w+r)\xi_j|, \quad w_j\epsilon_2 = |B+(w+s)\eta_j|, \quad w_j\epsilon_3 = |C+(w+t)\zeta_j|,$$

and it follows from the results above that at least one of the three products $\epsilon_1\epsilon_2$, $\epsilon_2\epsilon_3$, $\epsilon_3\epsilon_1$ is of magnitude $M.2^{-4[j-2]}$ for sufficiently large values of j. But, under the same condition,

$$M.2^{-4[j-2]} > 2^{-6[j-2]} > f(2^{[j-1]-[j-2]}) \geqq f(|w|).$$

Hence, when j exceeds a certain number, the condition (2) is not fulfilled. That is, the values of w satisfying this condition are limited.

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[1] Throughout this paper the term "integer" means "a positive or negative integer or zero," when it is not specially defined.

[2] Kronecker, L., *Monatsberichte Kgl. Preuss. Aka. Wiss.*, 1884 (1179ff., 1271ff.); *Werke*, 3^{I}, Leipzig, 1899 (49–109).

[3] See Dickson, L. E., *History of the Theory of Numbers*, **2**; *Carnegie Institution, Wash.*, 1920 (93–99), for references.

A PHYSICAL BASIS FOR EPIDEMIOLOGY

By Simon Flexner and Harold L. Amoss

Laboratories of the Rockefeller Institute for Medical Research

Read before the Academy, April 26, 1921

In the autumn of 1918 there swept through one of the mouse breeding rooms of the Rockefeller Institute a destructive epidemic of mouse typhoid—an infection of mice with a bacillus of the enteritidis group of organisms to which the name of *Bacillus typhi murium* has been given. The history of the epidemic is instructive: the original mice of the population, numbering about 3000 at the time of the epidemic, came from a breeder in Massachusetts and had been purchased some time before and moved *en masse* to the Rockefeller Institute. In the meantime many new mice had been born of this stock, and many of the original mice had died or been employed for experiment, so that only a small residue of the original population remained. In other words, the epidemic of mouse typhoid arose among chiefly a new stock of mice, the offspring of an old stock believed on good grounds to have passed through previous outbreaks of the disease.

There is still another reason for supposing that the epidemic arose from within and was not imported from without this stock. Besides the breeding room for the population accruing from the purchased mice, a second breeding room not far away is maintained for mice bred at the Institute from perfectly healthy stock. This second breeding station has been developed from small beginnings and now has an average mouse population of 3000. The personnel in charge of each breeding room is distinct and does not mingle. During the period of many months in which the epidemic continued, rose and fell in the one room, no death from mouse typhoid occurred in the other.

Our attention has long been directed to the phenomena of the epidemic prevalence of disease, and since the early years of the founding of the Rockefeller Institute opportunity was presented to observe in succession epidemics of meningitis, poliomyelitis, influenza, and latterly of encephalitis. The waves of these diseases, which have swept over the world since 1904, have aroused profound interest in the facts of epidemiology.

We seized, therefore, upon this outbreak of mouse typhoid in order to study experimentally an epidemic disease among small laboratory animals which could be assembled and observed in fairly large numbers. The epidemic referred to has supplied the cultures of the *Bacillus typhi murium* for the experiment and certain data of a statistical nature with which to compare our experimental results. The healthy stock of mice also referred to provided an unexceptionable material with which to attempt the production of an intentional epidemic.

The conditions of the experiment were simple. A kind of mouse village was set up and an isolated room away from all other animals was selected in which shelves were erected and the cages placed. The latter may be taken for streets and houses in an ordinary village. Each cage contained five healthy mice. The original population, later increased from time to time by the introduction of fresh increments, was 100. The epidemic was started by feeding with a virulent culture of *Bacillus typhi murium* ten mice placed in two cages midway of the other cages. The incidental contact between the culture fed mice and the others was secured by the attendant who fed the animals and periodically cleaned the cages. A spot map was kept in order to follow the events.

The preliminary feeding experiments led to the death from mouse typhoid of eight of the ten hand-fed and seven contact mice distributed in as many cages. No epidemic in the real sense ensued. The fatal outbreak of mouse typhoid which arose partook of the nature of sporadic instances of the disease.

When the conditions as regards death from the infection had become stationary (at about the end of 30 days), a fresh addition of 200 mice contained in cages of five each was brought into the village. The effect of this addition was striking: after a lapse of about five days the new mice began to die of mouse typhoid, and five to ten days later the old mice (original population) also began to succumb. The total losses in this outbreak were 70 per cent, and the number of cages involved all, or 100 per cent. The epidemic endured about four weeks and then subsided, although an occasional death still occurred at intervals.

When a second equilibrium had been established, another addition of normal or healthy stock was introduced into the village. The succession of events was about the same as that just described. However, the peak of the epidemic was less high, the total percentage of deaths being 5 per cent, although 100 per cent of the cages was again attacked. After an interval, another equilibrium was reached, when still another addition of normal mice brought about a recrudescence, passing as in the other examples from the healthy new population to the exposed old population, with however a total percentage of deaths below the others.

These fluctuations or movements of the epidemic consisting of a series of waves have been repeated now about ten times. The general character or type of the wave or curve is always the same, although the height or peak as determined by the percentage of mortality varies. In every instance the mortality rate among the old mice became finally about the same as among the new.

The experiments are followed in two main ways: according to the mortality, controlled of course by bacteriological examination of the animals succumbing, and according to the living carriers of the *Bacillus typhi murium* produced. The feces and the urine of living mice are tested for the bacillus, and in general it may be stated that the percentage of carriers stands in inverse ratio to the mortality. In other words, high death rate means a low carrier rate and *vice versa*. An obvious, although by no means safe, deduction from this observation would be that the carriers have all recovered from non-lethal infection and hence are rendered resistant by an acquired immunity, and that the production of carriers is nature's way of limiting the epidemic and of bringing about the state of equilibrium described.

The main results of the study presented may be viewed as the mere beginnings of an undertaking to follow artificially reproduced epidemics among animals, which not only should simulate the epidemics naturally or spontaneously arising, as we say, among men and animals, but which being controllable may be made to yield up some of the underlying conditions affecting the movements among those naturally occurring epidemic outbreaks which we represent graphically in the form of curves, but which in relation to cause and effect we still are so largely ignorant.

The impulse to use epidemic diseases among laboratory animals to help solve the problem of epidemics among man has been felt by other experimenters. In particular Topley[1] in London has independently made use also of the mouse and a bacillus of the enteritidis group with which to conduct experiments, and the reports he has already published, in spite of quite fundamental differences of procedure, are readily comparable with those we have obtained.

As a matter of practical importance perhaps the epidemic diseases affecting the respiratory tract may be more important than those in which the portal of entry into the body of the infectious microbe is by way of the digestive tube. The latter class has indeed been subjected to far greater control by sanitary and other measures than the former. But in endeavoring to work out the underlying principles ultimately to be assembled into the "laws" of epidemics, information from many sources will be required. It is part of our object, however, to report later on experimental epidemiological studies now in progress with a disease of laboratory animals in which the infecting bacillus thrives on the respiratory mucous membrane. It is obvious that the two chief factors involved in epidemiological investigations, namely the host and the microbe, are both highly intricate. Any comprehensive study will include both, and to the extent to which it is enlightening will account for the effects of the fluctuation of one on the response of the other, the two conditions having been correlated providing the data on which our ultimate understanding of epidemics will come to rest.

[1] *J. Hygiene,* **19,** 1921 (349).

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THE GROWTH AND DECAY OF PHOTO-THERMIONIC CURRENTS FROM OXIDE COATED FILAMENTS

By H. D. Arnold and Herbert E. Ives

Research Laboratories of the American Telephone & Telegraph Company and Western Electric Company, Inc.

Communicated by F. B. Jewett, October 18, 1921

T. W. Case[1] and E. Merritt[2] have recently described experiments in which the oxide coated filament of an audion is subjected to illumination, with the result of increasing the current between filament and plate. In the experiments of Prof. Merritt this added space current was shown to be principally due to short wave-length radiation, and was explained as a true photo-electric effect, that is, the direct emission of electrons due to illumination. He found the added current to be dependent upon the filament temperature. This might be taken to indicate that the emission of photo-electrons is a function of the temperature of the surface, which disagrees with conclusions drawn from experiments upon other photo-electrically active materials.

In some recent experiments we have found this effect to have characteristics sufficiently different from those of the true photo-electric effect to raise doubts as to the identity of the two phenomena. If they are not identical then the new effect is not necessarily evidence for a dependence of the photo-electric effect on temperature.

The experiments here reported have to do with the rate of growth and decay of the light induced additions to the space currents. These were recorded on a string galvanometer through a distortionless amplifier system. In every case voltages were high enough to carry across all electrons liberated. A carbon arc lamp with glass condensing lens was used as a light source, and between the condenser and the oxide coated filament was interposed either a deep red or a deep blue glass (transmitting no red), selected so that the light transmitted by each gave added space currents of the same order of magnitude.

With the deep red glass interposed the added space currents obtained upon brief illumination with various heating currents through the filament are shown in figure 1. The effect increases regularly with temperature, but takes a considerable time to rise and fall at the beginning and end of the period of illumination. The times of growth and decay depend upon the magnitude of the current, and the curves have the general appearance of heating and cooling curves. We believe this red light effect to be chiefly due to the heating of the filament by the incident radiation. Support of this belief is given by figure 2, in which the same increase of space current is caused, first (a) by red light, and second (b) by a sudden brief increase of filament heating current. It will be seen that the two curves are similar. We have also found that the added space current varies with filament current in the way that it would for equal added filament-power increments; and that the change of resistance of the filament is approximately the same whether the space current increment is due to red light or filament current.

With the blue glass, the energy transmission, as measured by a thermocouple, was only a few per cent that of the red glass, although for a chosen filament temperature near the middle of the available range both transmitted radiations gave the same added space current. The heating effect of the blue radiation is therefore negligible, and the added space current may be considered as a true light effect. String galvanometer records of the added space current due to brief blue illumination, for various filament currents (temperatures) are reproduced in figure 3. It will be seen at once that the added space currents exhibit a behavior essentially different from those due to red light or heating. Whereas with red light the current continually increases with temperature, with blue light the added current at first increases and then starts to decrease. At low temperatures the growth and decay are exceedingly slow. As the temperature is raised the rates of growth and decay increase until at the highest temperature studied the response appears practically instantaneous.

It is the existence of this slow response and its variation with temperature that we believe differentiate this effect from the true photo-electric effect, which, as far as previously known (and in agreement with tests made by us on a potassium cell) is instantaneous at all temperatures.

The true nature of the increased electronic emission under blue illumination is not decided by our experiments, but several possibilities suggest themselves. It may be a light induced increase in the number of electrons available for thermionic emission; or it may be a light induced change

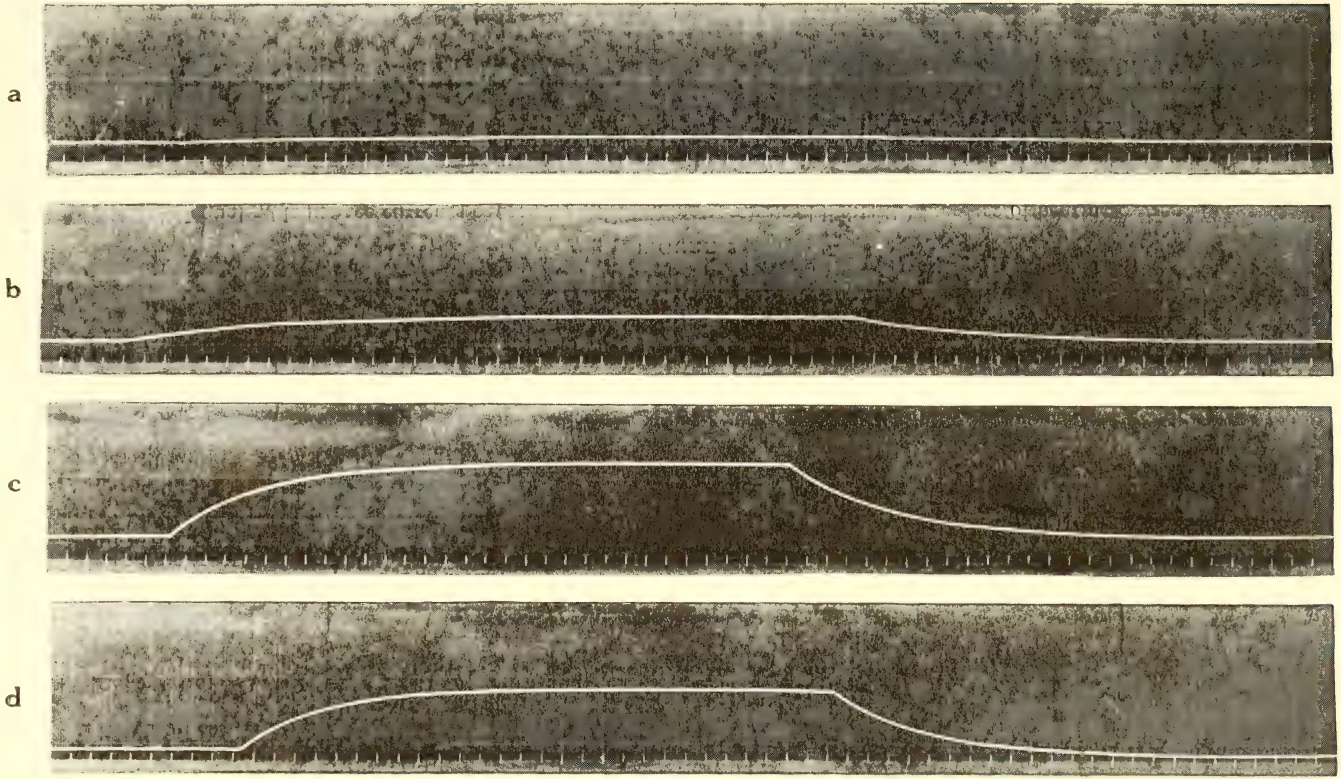

FIGURE 1

Added space currents due to illumination of audion filament by red light. (a) Filament heating current = .65 amp.; (b) filament heating current = .75 amp.; (c) filament heating current = .85 amp.; (d) filament heating current = .95 amp.; (sensibility of galvanometer reduced to half.) Larger time divisions = 0.1 second.

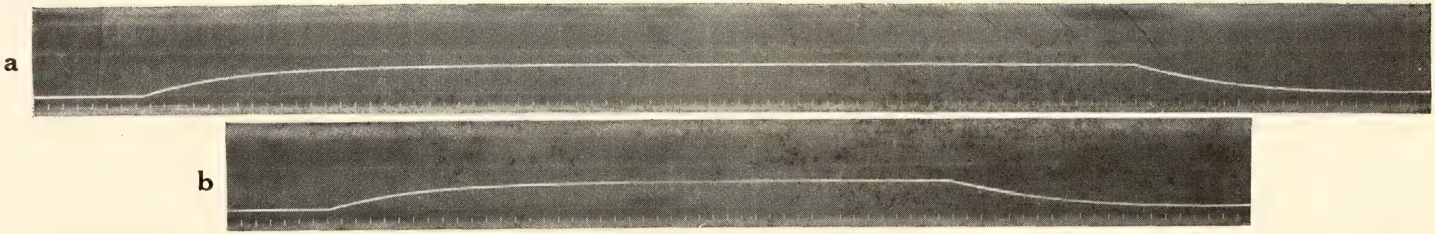

FIGURE 2

Equal added space currents caused by (a) illumination by red light; (b) increase of heating current.

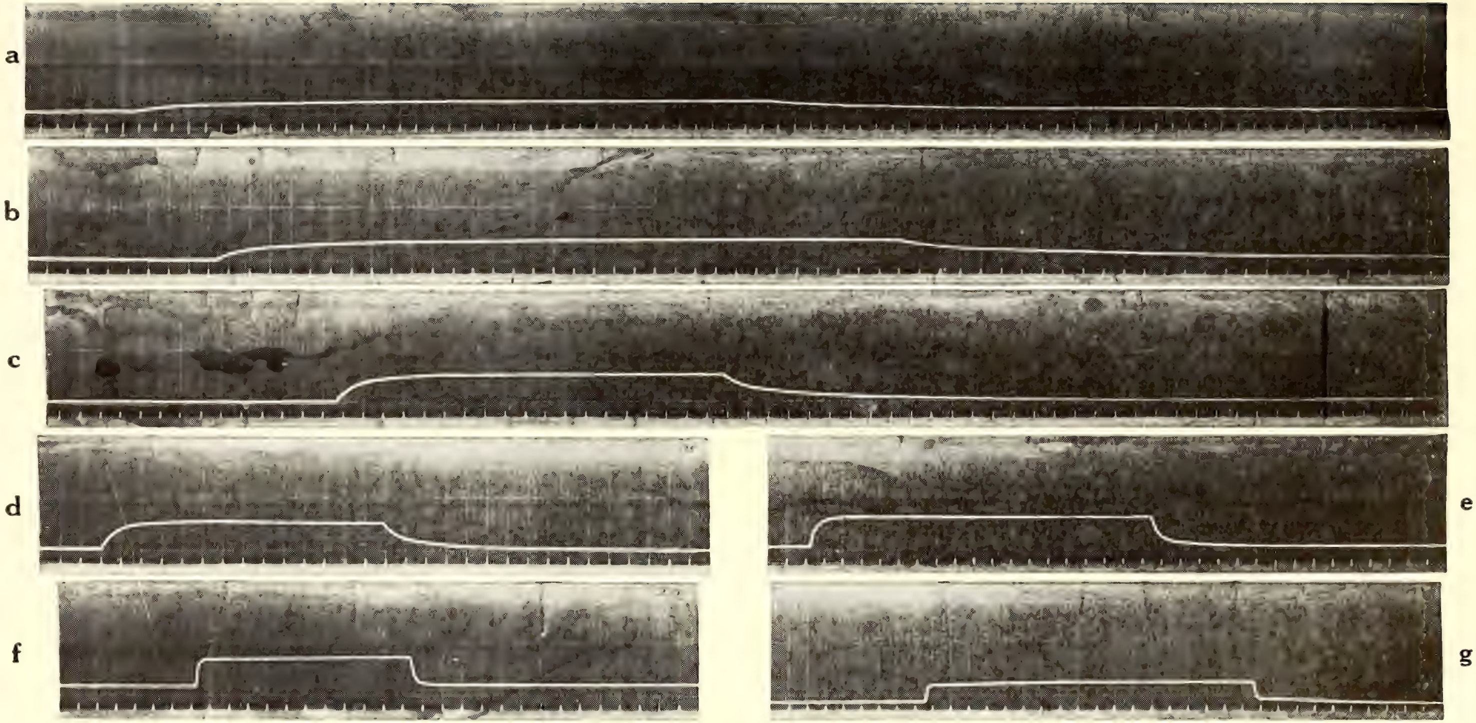

FIGURE 3

Added space currents due to illumination of audion filament by blue light. Filament heating currents = (a) .60 amp.; (b) .64 amp.; (c) .70 amp.; (d) .75 amp.; (e) .80 amp.; (f) .87 amp.; (g) .94 amp.
Filament temperature range 500° C. to 800° C. Larger time divisions = 0.1 second.

in the work function of the surface, so that more of the electrons present may escape;[3] or it may be a light induced change in the chemical nature of the surface. In any case there must be ascribed to the light produced change a temperature coefficient of speed of response. Some clue to the nature of the phenomenon may perhaps be obtained from a study of the growth and decay curves. They are not simple exponential curves such as would represent the charging and discharging of a condenser. If the reciprocal of the square root of the decaying current increment is plotted against time, the plots are very nearly straight lines for all temperatures (law of phosphorescence decay). Attention may be called to the fact that the growth and decay of current represented in the curves in figure 3 exhibiting slow response at low temperatures and quick response at high, are quite similar to the growth and decay of current in selenium under illumination, which likewise show a great variation of speed with temperature. This analogy suggests that the cause of the light effect in the oxide coated filament may be closely related to that which gives selenium its photo-sensitive properties. It should also be noted that the change of speed of response with temperature would be expected if one assumed the effect to be due to a chemical reaction.

In view of the data presented it is believed that this photo-effect in thermionic filaments cannot safely be considered as evidence for a variation of the true photo-electric effect with temperature. We suggest the use of the term "photo-thermionic emission."

[1] T. W. Case, *Physic. Rev.*, **17**, 3, p. 398.

[2] E. Merritt, *Ibid.*, **17**, 4, p. 525.

[3] Koppius (*Ibid.*, Mar. 1921, p. 395) found with oxide coated filaments a decrease in the work function with increasing temperatures.

GROUP OF ISOMORPHISMS OF A TRANSITIVE SUBSTITUTION GROUP

BY G. A. MILLER

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS

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Let G represent any transitive substitution group of degree n. It is convenient to divide the possible automorphisms of G into three categories. First, those which can be obtained by transforming all the substitutions of G successively by each of its own substitutions. In this way the group of inner isomorphisms of G can be found and this group is known to be an invariant subgroup of the group of isomorphisms I of G. The second category consists of the automorphisms obtained by transforming G by substitutions on its own letters, which transform G into itself but are not found in G. It will be proved that these automorphisms together with the inner ones constitute all the possible automorphisms of G in which the subgroups composed separately of all the substitutions of G omitting a particular letter correspond to such subgroups.

The third category of automorphisms of G is composed of all those in which the subgroups composed of all the substitutions of G which omit a given letter correspond to subgroups of degree n. Such automorphisms are not always possible. In fact it may happen that I is simply isomorphic with the group of inner isomorphisms of G. In this case only the first category of automorphisms actually exists. A necessary and sufficient condition that the first category of automorphisms reduces to the identity is that G is abelian.

To prove the theorem noted at the end of the first paragraph we may consider any possible automorphism of G in which the subgroup G_1 composed of all the substitutions of G which omit a given letter a corresponds to itself while some of the substitutions of this subgroup do not necessarily correspond to themselves. In this automorphism every substitution of G which involves the letter a must correspond to such a substitution, and we shall suppose that the substitutions are represented in such a way that this letter appears first.

There are g/n, g being the order of G, substitutions of G in which the letter a is followed by any other letter b. If in one of the corresponding substitutions under the said automorphisms a is followed by d then this must be the case in each of the g/n substitutions which correspond to the substitutions beginning with the letters ab. If this were not the case as regards some two such corresponding substitutions then the inverses of the former of the two given corresponding substitutions into the latter would give two corresponding substitutions of which the former would omit the letter b while the latter would involve the letter d. This is impossible since the subgroup composed of all the substitutions which omit b must correspond to the subgroup composed of all the substitutions which omit d in the automorphism under consideration as a result of the fact that a substitution beginning with ab corresponds to one beginning with ad.

From the preceding paragraph it results that the second letters in all of the corresponding substitutions involving the letter a in the said automorphism determine a single substitution t on the letters of G. This substitution does not involve the letter a. It transforms G into a simply isomorphic group G' such that in the corresponding substitutions involving a the second letters are exactly the same as they are in the second group of the automorphism under consideration. If G were not identical with this second group, as a substitution group, it would be possible to establish a simple isomorphism between G and another substitution group G' such that every substitution involving a would correspond to such a substitution and that the second letter in all these corresponding substitutions would be the same in every case.

In this simple isomorphism every two corresponding substitutions would be identical. For, if S_1, S_1' are any two corresponding substitutions such that in S_1 the letter e is followed by the letter f, and if S_2, S_2' are two other corresponding substitutions in which the letter a is followed by e then S_1 is obtained by multiplying the inverse of S_2 into one of the g/n substitutions in which a is followed by f. Hence it results that in S_1' the letter e is also followed by the f. That is, S_1 and S_1' are identical. In this proof it was assumed that e is not a and that f is not a. If e were a then a would be replaced by f in S_1' by hypothesis. If f were a then in the inverse of S_1 the letter a would be replaced by e and hence in the inverse of S_1' the letter a would be replaced by e. Therefore the theorem again requires no further proof. Hence it has been established that G and G' are identical, and that the automorphism under consideration can be effected by transformed G by the substitution t which therefore transforms G into itself.

If the subgroup G_1 had corresponded in the given automorphism to a conjugate of G_1 but different from G_1 there would be a substitution in G which would transform G_1 into this conjugate. The inverse of this substitution into t would transform G into itself and effect the automorphism in question. It has therefore been proved that *every automorphism of G in which the subgroups composed of all the substitutions of G which omit a given letter correspond to such subgroups is effected by substitutions of the largest group on the letters of G which contains G invariantly.* As a particular case of this theorem we have the well known result that when G is regular it is transformed into all its possible automorphisms under its holomorph.

From the theorem proved in the preceding paragraph it is easy to deduce a method for finding the order of the group of isomorphisms of the transitive substitution group G. If G_1 is of degree $n - \alpha$ the largest group on the letters of G which contains G invariantly involves exactly α substitutions which are commutative with every substitution of G and hence this largest group transforms the substitutions of G according to a group whose order is the order of this largest group divided by α. This group of transformation is simply isomorphic with the quotient group of this largest group with respect to its invariant subgroup composed of the α substitutions which are commutative with every substitution of G. The order of the I of G is therefore equal to the order of this quotient group multiplied by the number of the different sets of conjugate subgroups of G which are such that each set involves a subgroup which corresponds to G_1 in some possible automorphism of G.

A necessary and sufficient condition that G contains more than one such set of conjugate subgroups is that it contains subgroups of degree n and of index n with respect to which it can be represented as a transitive substitution group which is identical with G. In particular, *if a given abstract group is simply isomorphic with only one group in a complete list of transitive groups of degree n then this transitive group has outer isomorphisms if it contains a subgroup of degree n and of index n which is non-invariant and does not involve any invariant subgroup of the entire group besides the identity.* From this special but useful theorem it follows directly that the symmetric group of degree 6 has outer isomorphisms. Various writers established this fact by somewhat laborious special methods.[1] It also follows directly from this theorem that the largest imprimitive groups on six letters which involve 2 and 3 systems of imprimitivity, respectively, have isomorphisms which cannot be obtained by transforming these groups by substitutions on their own letters.

While the group of inner isomorphisms is an invariant subgroup of I whenever it does not coincide with I it should not be inferred that the subgroup of I which corresponds to all the automorphisms of G which can be obtained by transforming G by substitutions on its own letters is always an invariant subgroup of I. In fact, this is not the case when G is the generalized dihedral group of order 16 involving the abelian group of type (2, 1) represented as a transitive group of degree 8. When G admits automorphisms in which G_1 corresponds to a subgroup of degree n the conjugates of G_1 are transformed under I according to an imprimitive substitution group. One of the systems of imprimitivity of this group is composed of the conjugates of G_1 under G.

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[1] Cf. O. Hölder, *Mathematische Annalen*, **46**, 1895 (345); W. Burnside, *Theory of Groups of Finite Order*, 1911, p. 209.

EINSTEIN STATIC FIELDS ADMITTING A GROUP G_2 OF CONTINUOUS TRANSFORMATIONS INTO THEMSELVES

By L. P. Eisenhart

Department of Mathematics, Princeton University

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1. For static phenomena in the Einstein theory the linear element of the space-time continuum can be taken in the form $V^2dx_0^2-ds^2$, where

$$ds^2 = \Sigma\, a_{ik}\, dx_i\, dx_k \; (i, k = 1, 2, 3) \qquad (1)$$

is the linear element of the physical space S, and the functions V and a_{ik} are independent of x_0, the coördinate of time. In this paper we determine the functions V and a_{ik} satisfying the Einstein equations $B_{ik} = O$ and such that the space S admits a continuous group G_2 of transformations into itself. Bianchi[1] has shown that any 3-space admitting such a group is of one of two types:

$$\text{(I)} \qquad ds^2 = dx_1^2 + \alpha dx_2^2 + 2\beta dx_2 dx_3 + \gamma dx_3^2,$$

α, β, γ being functions of x_1 alone, the operators of the group being

$$X_1 = \partial/\partial x_2, \; X_2 = \partial/\partial x_3;$$

$$\text{(II)} \quad ds^2 = dx_1^2 + \alpha dx_2^2 + 2(\beta - \alpha x_2)\, dx_2 dx_3 + (\alpha x_2^2 - 2\beta x_2 + \gamma) dx_3^2,$$

α, β, γ being functions of x_1 alone, the operators of the group being

$$X_1 = \partial/\partial x_3, \quad X_2 = e^{x_3}\partial/\partial x_2.$$

2. By definition

$$B_{ik} = \sum_0^3{}_h \left\{ih, hk\right\} = \sum_0^3{}_h \left[\frac{\partial}{\partial x_k}\left\{\begin{matrix} ih \\ h \end{matrix}\right\} - \frac{\partial}{\partial x_h}\left\{\begin{matrix} ik \\ h \end{matrix}\right\} + \right.$$

$$\left. \sum_0^3{}_e \left(\left\{\begin{matrix} ih \\ e \end{matrix}\right\}\left\{\begin{matrix} ke \\ h \end{matrix}\right\} - \left\{\begin{matrix} lh \\ h \end{matrix}\right\}\left\{\begin{matrix} ik \\ e \end{matrix}\right\}\right)\right], \tag{2}$$

where the Christoffel quantities are calculated with respect to the linear element of the space-time continuum. When this linear element is taken in the form $V^2dx_0^2 - ds^2$ as in § 1, the equations $B_{ik} = 0$ reduce to the six equations[2]

$$\frac{\partial^2 V}{\partial x_i \partial x_k} = \sum_1^3{}_e \left\{\begin{matrix} ik \\ e \end{matrix}\right\} \frac{\partial V}{\partial x_e} - \bar{B}_{ik} V \; (i, k = 1, 2, 3), \tag{3}$$

and

$$\sum_{i,k} a^{ik} \bar{B}_{ik} = 0, \tag{4}$$

where a^{ik} is the cofactor of a_{ik} in the determinant of the quantities a_{ik} divided by this determinant; the Christoffel symbols are calculated with respect to (1), and

$$\bar{B}_{ik} = \sum_1^3{}_h \left\{ih, hk\right\} = \sum_1^3{}_h \sum_1^3{}_l a^{hl}\left(il, hk\right). \tag{5}$$

By assuming that S is referred to a triply-orthogonal system, so that the linear element may be written $ds^2 = \sum_1^3{}_i\, a_i dx_i^2$ (which is possible in any 3-space[3]), and making use of the relations

$$(il, hk) = -(li, hk) = -(il, kh) = (li, kh),$$

we find that when $\bar{B}_{ik} = 0$, then all the functions (il, hk) are equal to zero. This being the condition that the space S is euclidean, we have the theorem:

A necessary and sufficient condition that a 3-space be euclidean is that the functions $\bar{B}_{ik} = 0$.

3. If we put

$$\delta = \alpha\gamma - \beta^2,\ A = \beta'\gamma - \beta\gamma',\ B = \gamma'\alpha - \gamma\alpha',\ C = \alpha'\beta - \alpha\beta',$$

(where primes indicate differentiation with respect to x_1) we find that all of the Christoffel quantities for the form (I) vanish except the following:

$$\begin{Bmatrix}22\\1\end{Bmatrix} = -\frac{\alpha'}{2},\ \begin{Bmatrix}23\\1\end{Bmatrix} = -\frac{\beta'}{2},\ \begin{Bmatrix}33\\1\end{Bmatrix} = -\frac{\gamma'}{2},\ \begin{Bmatrix}13\\2\end{Bmatrix} = \frac{A}{2\delta},$$

$$\begin{Bmatrix}12\\2\end{Bmatrix} = \frac{1}{4\delta}\left(\delta' - B\right),\ \begin{Bmatrix}12\\3\end{Bmatrix} = -\frac{C}{2\delta},\ \begin{Bmatrix}13\\3\end{Bmatrix} = \frac{1}{4\delta}\left(\delta' + B\right).$$

Also we find

$$\overline{B}_{11} = \frac{\delta''}{2\delta} - \frac{1}{4}\frac{\delta'^2}{\delta^2} - \frac{1}{2\delta}\left(\alpha'\gamma' - \beta'^2\right),\ \overline{B}_{12} = \overline{B}_{13} = 0,$$

$$\overline{B}_{22} = \frac{1}{2}\alpha'' - \frac{\alpha'}{4}\frac{\delta'}{\delta} + \frac{\alpha}{2\delta}\left(\alpha'\gamma' - \beta'^2\right),$$

$$\overline{B}_{23} = \frac{1}{2}\beta'' - \frac{\beta'}{4}\frac{\delta'}{\delta} + \frac{\beta}{2\delta}\left(\alpha'\gamma' - \beta'^2\right),$$

$$\overline{B}_{33} = \frac{1}{2}\gamma'' - \frac{\gamma'}{4}\frac{\delta'}{\delta} + \frac{\gamma}{2\delta}\left(\alpha'\gamma' - \beta'^2\right).$$

Equation (4) reduces to

$$2\,\alpha\,\gamma'' + 2\,\alpha''\,\gamma + 3\left(\alpha'\,\gamma' - \beta'^2\right) - 4\,\beta\,\beta'' - \frac{\delta'^2}{\delta} = 0. \qquad (6)$$

The problem reduces to the integration of (3), when α, β, γ are subject to the condition (6). We separate the discussion into the four cases, when V is independent of x_2 and x_3; of either x_2 or x_3; of x_1; involves all three variables. Expressing the conditions of integrability of (3) we are led to linear partial differential equations of the first order and these are ultimately solved. The first and second cases are the only ones giving solutions; in the course of the investigation it is shown that in both cases a change of variables can be made so that $\beta = 0$. The solutions of the first type are:

$$\alpha = x_1^{\frac{2(k+1)}{k^2+k+1}},\ \gamma = x_1^{\frac{2k(k+1)}{k^2+k+1}},\ V = x_1^{-\frac{k}{k^2+k+1}}, \qquad (7)$$

where k is any constant. Solutions of this kind have been found by Kasner.[4] For the second case the linear element of S may be written

$$ds^2 = \frac{d\alpha^2}{b\,\alpha + a\,\alpha^{1/2}} + \alpha\,dx_2^2 + \left(b + a\,\alpha^{-\frac{1}{2}}\right)d\,x_3^2, \qquad (8)$$

and

$$\mathrm{V} = \alpha^{1/2}\varphi(x_2), \qquad \frac{d^2\varphi}{dx_2^2} + b\varphi = 0, \qquad (9)$$

where a and b are constants. This is the solution found by Levi-Civita[5] and called the *quadrantal* solution.

4. For the linear element (II) the expressions for the Christoffel symbols which are different from zero are:

$$\begin{Bmatrix}22\\1\end{Bmatrix} = -\frac{\alpha'}{2}, \begin{Bmatrix}23\\1\end{Bmatrix} = \frac{1}{2}\left(\alpha' x_2 - \beta'\right),$$

$$\begin{Bmatrix}33\\1\end{Bmatrix} = -\frac{1}{2}\left(\alpha' x_2^2 - 2\beta' x_2 + \gamma'\right),$$

$$\begin{Bmatrix}12\\2\end{Bmatrix} = \frac{1}{4\delta}\left(-2Cx_2 - B + \delta'\right), \begin{Bmatrix}13\\2\end{Bmatrix} = \frac{1}{2\delta}\left(Cx_2^2 + Bx_2 + A\right),$$

$$\begin{Bmatrix}22\\2\end{Bmatrix} = -\frac{\alpha}{\delta}\left(\alpha x_2 - \beta\right), \begin{Bmatrix}23\\2\end{Bmatrix} = \frac{(\alpha x_2 - \beta)^2}{\delta},$$

$$\begin{Bmatrix}33\\2\end{Bmatrix} = \frac{\beta - \alpha x_2}{\delta}\left(\alpha x_2^2 - 2\beta x_2 + \gamma\right),$$

$$\begin{Bmatrix}12\\3\end{Bmatrix} = -\frac{C}{2\delta}, \begin{Bmatrix}13\\3\end{Bmatrix} = \frac{1}{4\delta}\left(2Cx_2 + B + \delta'\right), \begin{Bmatrix}22\\3\end{Bmatrix} = -\frac{\alpha^2}{\delta},$$

$$\begin{Bmatrix}23\\3\end{Bmatrix} = \frac{\alpha(\alpha x_2 - \beta)}{\delta}, \begin{Bmatrix}33\\3\end{Bmatrix} = -\frac{(\alpha x_2 - \beta)^2}{\delta}.$$

The expressions for the functions $\overline{B}_{ik}$ are

$$\overline{B}_{11} = \frac{\delta''}{2\delta} - \frac{1}{4}\frac{\delta'^2}{\delta^2} - \frac{1}{2\delta}\left(\alpha'\gamma' - \beta'^2\right), \quad \overline{B}_{12} = \frac{C}{\delta},$$

$$\overline{B}_{13} = -\frac{1}{2\delta}\left(2Cx_2 + B\right),$$

$$\overline{B}_{22} = L, \overline{B}_{23} = M - x_2 L, \quad \overline{B}_{33} = Lx_2^2 - 2Mx_2 + N,$$

where

$$L = \frac{\alpha''}{2} - \frac{\alpha'\delta'}{4\delta} + \frac{\alpha}{2\delta}\left(\alpha'\gamma' - \beta^2\right) + \frac{\alpha^2}{\delta},$$

$$M = \frac{\beta''}{2} - \frac{\beta'}{4}\frac{\delta'}{\delta} + \frac{\beta}{2\delta}\left(\alpha'\gamma' - \beta'^2\right) + \frac{\alpha\beta}{\delta},$$

$$N = \frac{\gamma''}{2} - \frac{\gamma'}{4}\frac{\delta'}{\delta} + \frac{\gamma}{2\delta}\left(\alpha'\gamma' - \beta'^2\right) + \frac{\alpha\gamma}{\delta}.$$

The condition (4) is

$$2\delta'' - \frac{\delta'^2}{\delta} - \left(\alpha'\gamma' - \beta'^2\right) + 4\alpha = 0. \tag{10}$$

We consider the system of equations (3) as in § 3, taking up separately the five cases where V is independent of x_2 and x_3; of x_2; of x_1; of x_3; involves all three variables. Only the first two cases lead to solutions.

For the first case we have the linear element of S

$$ds^2 = \frac{d\alpha^2}{4(a\,\alpha^{1/2} - \alpha)} + \alpha\, dx_2^2 - 2\,\alpha x_2\, dx_2\, dx_3 + \alpha\,(x_2^2 + 1)\, dx_3^2, \tag{11}$$

and

$$V = \sqrt{a\,\alpha^{-1/2} - 1},$$

where a is a constant. This is the *longitudinal* solution obtained by Levi-Civita.[5]

For V independent of x_2 two cases arise, according as β can be made equal to zero by a transformation of coördinates, or not.

When $\beta = o$, the functions α and γ must satisfy the three equations

$$\begin{gathered} \alpha'' + \left(1 + \frac{1}{2k}\right)\frac{\alpha'\gamma'}{\gamma} - \left(1 + \frac{1}{k}\right)\frac{\alpha'^2}{2\alpha} + \frac{2\alpha}{\gamma}\left(1 + k\right) = 0, \\ \gamma'' + \left(1 - \frac{1}{k}\right)\frac{\alpha'\gamma'}{2\alpha} + \frac{1}{2k}\frac{\gamma'^2}{\gamma} + 2\left(1 + k^2\right) = 0, \qquad (12) \\ 2\,\alpha'\gamma' - \frac{\alpha'^2\gamma}{\alpha}\frac{1}{k} + \left(1 + \frac{1}{k}\right)\frac{\gamma'^2\alpha}{\gamma} + 4\,\alpha\left(k^2 + k + 1\right) = 0, \end{gathered}$$

where k is any constant. For $k = 1$,

$$\gamma\,\gamma'^2 + 4\,\gamma^2 = 4\,a, \quad \frac{\alpha}{\gamma} = b\left(\frac{\sqrt{a} + \sqrt{a - \gamma^2}}{\gamma}\right)^{\pm\sqrt{3}}, \quad V = e^{x_3}\sqrt{\alpha\,\gamma} \qquad (13)$$

where a and b are constants. Consequently,

$$ds^2 = \frac{4(a - \gamma^2)}{\gamma}\,d\gamma^2 + \alpha\,dx_2^2 - 2\,\alpha\,x_2\,dx_2\,dx_3 + \left(\alpha\,x_2^2 + \gamma\right)dx_3^2. \qquad (14)$$

When $k \neq 1$, we solve the second of (12) for α'/α and substitute in the third of (12). The first integral of the resulting equation is

$$\frac{\left(\gamma' + \sqrt{\gamma'^2 + 4\,k\,\gamma}\right)^b}{\left[(1 + k^2)\sqrt{\gamma'^2 + 4\,k\,\gamma} + b\,\gamma'\right]^{1 + k^2}} = a\,\gamma^{k + \frac{b}{2}}, \qquad (15)$$

where a is an arbitrary constant and $b = \pm(1 - k)\sqrt{1 + k + k^2}$. Then we have

$$\begin{gathered} \alpha^{\frac{1}{k} - 1} = \gamma^{\frac{1'}{k}}\left[(1 + k^2)\sqrt{\gamma'_2 + 4\,k\,\gamma} + b\,\gamma'\right]^2, \\ V = e^{kx_3}\sqrt{\gamma}\left(\frac{\gamma}{\alpha}\right)^{\frac{1}{2k}}. \end{gathered} \qquad (16)$$

When $\beta \neq 0$, we have

$$V = \delta^{1/4}\,e^{-2x_3}, \quad \delta = \alpha\,\gamma - \beta^2, \qquad (17)$$

with the following conditions upon α, β and γ:

$$\begin{gathered} \alpha'' - \frac{\alpha'\delta'}{4\delta} - \frac{\alpha}{2}\frac{\delta'^2}{\delta^2} - 14\frac{\alpha^2}{\delta} = 0, \\ \beta'\alpha - \beta\alpha' = k\delta^{1/4}, \qquad (18) \\ \delta'^2 + 2\delta\,(\alpha'\gamma' - \beta'^2) + 24\,\alpha\,\delta = 0, \end{gathered}$$

where k denotes an arbitrary constant.

Now (10) reduces to

$$\delta'' - \frac{1}{4}\frac{\delta'^2}{\delta} + 8\,\alpha = 0, \tag{19}$$

which is consistent with (18). Eliminating α from the first of (18) and (19), we get

$$\delta'''' - \frac{3}{4}\frac{\delta'\,\delta'''}{\delta} + \frac{5}{4}\frac{\delta''^2}{\delta} - \frac{21}{64}\frac{\delta'^4}{\delta^3} = 0.$$

If we introduce dependent and independent variables t and θ by

$$\delta = e^{4\int e^{\theta}\,dx_1}, \quad t = e^{-\theta}\frac{d\theta}{dx_1},$$

we get the equation

$$t\frac{d}{d\theta}\left(t\frac{dt}{d\theta}\right) + 7t^2\frac{dt}{d\theta} + 13\,t\frac{dt}{d\theta} + (2t+5)\,(3t+5)\,(t+3) = 0.$$

To the evident solutions of this equation, $t = -5/2$, $t = -5/3$, correspond the solutions:

$$\delta = 25\,x_1^{8/5}, \quad \alpha = -\,x_1^{-2/5}, \; \beta = b\,x_1^{9/5}, \; \gamma = -\,b^2 x_1^4 - 25\,x_1^2,$$

$$\delta = 25\,x_1^{12/5}, \; \alpha = -\,6\,x_1^{2/5}, \quad \beta = b\,x_1^{6/5}, \; \gamma = -\frac{1}{6}(b^2+25)\,x_1^2,$$

where b is an arbitrary constant. For $t = -3$, we find $\alpha = 0$, which is excluded since we assume that $\delta > 0$ in accordance with the theory.

If we put $t\,dt/d\theta = y$, and take y for dependent variable, the above equation may be replaced by

$$y\frac{dy}{dt} + y\left(7t+13\right) + \left(2t+5\right)\left(3t+5\right)\left(t+3\right) = 0.$$

When a solution of this equation is known, the corresponding functions α, β, γ can be found by quadratures.

5. Making use of the formulas of Bianchi[6] we find the following expressions for the principal curvatures of the spaces (7), (8), (11), (14)

$$-\underline{\overline{K}}_1 = (1+k)\,\underline{\overline{K}}_2 = \left(1+\frac{1}{k}\right)\underline{\overline{K}}_3 = \frac{k\,(k+1)^2}{(1+k+k^2)^2}\,\frac{1}{x_1^2}; \tag{7*}$$

$$\underline{\overline{K}}_1 = -\frac{\underline{\overline{K}}_2}{2} = \underline{\overline{K}}_3 = \frac{a}{2\,\alpha^{3/2}}; \tag{8*}$$

$$-\frac{\underline{\overline{K}}_1}{2} = \underline{\overline{K}}_2 = \underline{\overline{K}}_3 = \frac{a}{2\,\alpha^{3/2}}; \tag{11*}$$

$$\underline{\overline{K}}_1 = -\frac{a}{\gamma^3}(1+\sqrt{3}), \; \underline{\overline{K}}_2 = \frac{2a}{\gamma^3}, \; \underline{\overline{K}}_3 = \frac{-a}{\gamma^3}(1-\sqrt{3}). \tag{14*}$$

The principal curvatures for $k \pm 1$ in (12) can be obtained explicitly, but their forms are quite involved.

At each point of space the principal curvatures correspond to three directions, mutually perpendicular to one another. When the curves tangent to these directions are the curves of intersection of a triply-orthogonal system of surfaces, the space is called *normal* by Bianchi. All the spaces referred to above are normal. For the cases (7) and (8) the tangents to the curves of intersection x_i = const., x_j = const. are the principal directions.

For the cases of § 4 we put $x_2 = e^{\bar{x}_2 + x_3}$. Then for (11), the curves of intersection of surfaces x_1 = const., $\bar{x}_2$ = const., x_3 = const. have the principal directions. For (14) and the case $k \mp 1$, the principal directions are given by x_1 = const., x_3 = const., and the orthogonal systems of curves on $\bar{x}_2$ = const. defined by

$$\left(\alpha' - \frac{\alpha\gamma'}{\gamma}\right)dx_1^2 + 2\left[\alpha + \frac{1}{4}\alpha'\gamma' - \gamma\left(\frac{\alpha''}{2} - \frac{\alpha'^2}{4\alpha}\right)\right]dx_1\,dx_3 \\ - \gamma\left(\alpha' - \frac{\alpha\gamma'}{\gamma}\right)dx_3^2 = 0.$$

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[1] *Mem. Soc. Ital.*, 1896, p. 347.

[2] Levi-Civita, *Rend Lincei* (ser. 5), **26**, 1917, sem. 1 (460).

[3] Bianchi, *Lezioni*, **1**, 377; Cotton, *Ann. Fac. Toul.* (ser. 2), **1**, 1899 (410).

[4] *Science*, **54**, 1921 (305).

[5] *Rend. Lincei* (ser. 5), **27**, 1918, sem. 2 (350).

[6] *Lezioni*, **1**, 354.

GEOMETRIC ASPECTS OF THE ABELIAN MODULAR FUNCTIONS OF GENUS FOUR (II)

By Arthur B. Coble

Department of Mathematics, University of Illinois

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8. *The form* $\binom{3\ 2\ 2}{1\ 1\ 1}$.—This form, written symbolically as $(\rho z)\ (rx)(sy)$, where z is a point in S_3, x a point in S_2, and y a point in S_2', has 36 coefficients and therefore $35-15-8-8 = 4$ absolute projective constants. Points x, y determine a plane which becomes indeterminate for six pairs $x, y = p_i\,q_i$; ($i = 1, \ldots, 6$) which form associated six points. They are the double singular points of a Cremona transformation T of the fifth order between the planes S_x, S_y. A given plane u is determined by ∞^1 pairs x, y which lie, respectively, on the cubic curves, $(\rho\rho'\rho''u)\ (rx)\ (r'x)\ (r''x)\ (ss's'') = 0$, $(\rho\rho'\rho''u)\ (rr'r'')\ (sy)\ (s'y)\ (s''y) = 0$ These curves pass, respectively, through the six points p_i and the six points q_i. Thus the given form is associated with a general cubic surface, $(\rho z)\ (\rho' z)\ (\rho'' z)\ (rr'r'')\ (ss's'') = 0$, with an isolated double-six of lines and separated

line-sixes. The mapping of the surface from the planes S_x and S_y is given by the above systems of cubics.

9. *The form* $\left(\begin{smallmatrix}1&1&2&2\\1&1&1&1\end{smallmatrix}\right)$.—In the space figure just described we insert a quadric Q with generators t, τ. The point coördinates z can be replaced by bilinear expressions in t, τ and the form $\left(\begin{smallmatrix}3&2&2\\1&1&1\end{smallmatrix}\right)$ becomes a form $(p\tau)$ (πt) (rx) (sy) general of its type, with thirteen absolute constants corresponding to the four absolute constants of the above cubic surface and the nine additional constants for the inserted quadric. The quadric meets the cubic surface in a normal curve of genus 4 so that we have in space the figure of the normal sextic and a particular one of the ∞^4 cubic surfaces through it. In the planes S_x, S_y we have projectively general (with thirteen absolute constants) sextics of genus 4 with nodes at p_i and q_i, respectively, transforms of each other under T. The canonical adjoints of these sextics, with isolated g_1^3's, are obtained by substituting for u in 8 the proper bilinear expressions in t, τ; while a similar substitution for z in the equation of the cubic surface gives the equation on the quadric of the normal curve. We may therefore regard the general form $\left(\begin{smallmatrix}1&1&2&2\\1&1&1&1\end{smallmatrix}\right)$ as a definition of the projectively general plane sextic of genus 4.

10. *The forms* $F, \bar{F}$ *on the conic* $K(\tau)$. *Counter sextics.*—We mark the conic $K(\tau)$ on the plane S_x and plot with reference to it the two counter sextics $\bar{S}_2(t)$ and $S_2(t)$. For each there is a cusp locus of perspective cubics $\bar{G}C(\tau)$ and $GC(\tau)$. Similarly we mark on a plane S_y the conic $K(t)$ and plot with reference to it the two counter sectics $\bar{S}_1(\tau)$, $S_1(\tau)$ which are paired with the above counter sextics, and which have for cusp loci of perspective cubics the sextic curves $GC(t)$, $\bar{G}C(t)$, respectively.

We now consider the form $F = (\alpha t)^3\ (a\tau)^3 = 0$ where τ is a tangent of $K(\tau)$. For variable t we have ∞^1 triangles circumscribed about $K(\tau)$ whose vertices run over a sextic curve. If t determines τ_1, τ_2, τ_3 then the point t, τ_1, τ_2, of this sextic curve is birationally related to the solution t, τ_3 of $F = 0$. Hence the sextic curve is birationally equivalent to $GC(\tau)$ and as a result of the algebraic discussion of the next two sections we prove that this sextic curve is actually $GC(\tau)$. This amounts to the effective elimination of t from $(a\tau_1)^3\ (\alpha t)^3 = 0$, $(a\tau_2)^3\ (\alpha t)^3 = 0$ and the separation of the factor $(\tau_1\tau_2)^3$. Hence we have on the sextic $GC(\tau)$ two g_1^3's such that the ∞^1 t-triads are cusp triangles of perspective cubics of $S_2(t)$ and the ∞^1 τ-triads are triangles circumscribed to $K(\tau)$ for which the intersection of the lines joining each vertex to the contact of the opposite side is the point t of $\bar{S}_2(t)$. Of course a similar statement is true of any one of the four cusp loci.

If we polarize the form F into $(a\tau)$ $(a\tau_1)$ $(a\tau_2)$ (αt) (αt_1) (αt_2) and replace the pairs τ_1, τ_2; t_1, t_2 by points x; y referred in Darboux coördinates to the conics $K(\tau)$; $K(t)$ in Sx; Sy, respectively, we have a form (πx) $(d\tau)$ (δt) (ρy) of the type discussed in 9. Hence the sextics $GC(\tau)$ and $GC(t)$ are transforms of each other under the quintic Cremona transformation T but on the two the role of the t and τ triads with reference to the rational sextics and conics K is reversed. It may be shown that the projective peculiarity (equivalent to four conditions) of our birationally general sextics $\overline{G}C(\tau)$ and $GC(\tau)$ is that their τ triads envelop a conic $K(\tau)$. This same property belongs to the t-triads after transformation of $GC(\tau)$ by T (or $\overline{G}C(\tau)$ by the corresponding $\overline{T}$) into a sextic in the plane S_y.

We have mentioned in I 7 certain intersectional properties of the conic $\bar{b}$, the rational sextic $\bar{S}_2(t)$, and the cusp locus $\overline{G}C(\tau)$. Analogous results for the conic $K(\tau)$, the sextic $\bar{S}_2(t)$, and the locus $GC(\tau)$ are as follows. The conic $K(\tau)$ meets $\bar{S}_2(t)$ in 12 points whose parameters t on $\bar{S}_2(t)$ are the branch points of the function $\tau(t)$ determined by $F = 0$, and whose parameters τ on $K(\tau)$ are the branch point values of this function. The 12 residual function values τ furnish 12 common tangents of $K(\tau)$ and $GC(\tau)$, i.e., the tangents to $GC(\tau)$ at the 12 coincidences of $g_1^3(\tau)$ upon it touch $K(\tau)$. The remaining common tangents of $K(\tau)$ and $GC(\tau)$ are double tangents of $GC(\tau)$ whose parameters on $K(\tau)$ are the branch points of the function $t(\tau)$ in $F = 0$. The two further tangents to $K(\tau)$ from the contacts with $GC(\tau)$ of each double tangent meet in a point which is a coincidence of $g_1^3(t)$ on $GC(\tau)$.

11. *The reciprocity between the forms F, $\overline{F}$; algebraic discussion.*—We denote by A the determinant of the coefficients (without binomial factors) of the form F; the coefficients of $\overline{F}$ are then, to within numerical factors, three-row minors of A. The reciprocity between the forms F, $\overline{F}$ is then brought out by the fact that the covariant $\overline{F}$ formed for $\overline{F}$ as a ground form is again $F.A^2/3^4$. The invariant $(aA)^3\ (\alpha A)^3$ is $4\ A$. This reciprocity fails when A vanishes. Then the four cubics in t furnished by F are linearly dependent. When expressed in terms of three the coefficients in F are three cubics in τ, and the form $F = 0$ can be interpreted as the incidence condition of point τ on a rational *plane* cubic curve C_1 with line t of a rational *plane* cubic envelope C_2. In this case the form $\overline{F}$ factors into two cubics $(A\tau)^3.(At)^3$, the conjugate cubics of the rational curves C_1, C_2. Now all the (usually non-vanishing) odd transvectants of the double form $\overline{F}$ will vanish whence in the general case they contain the factor A and are of the second degree for F. From their degree and orders they can be identified at once with $^2/_9\ (b\tau)^4\ (\beta t)^4 \equiv (aa')\ (\alpha\alpha')\ (a\tau)^2\ (a'\tau)^2\ (\alpha t)^2\ (\alpha' t)^2$; $8\,\delta \equiv (aa')^3\ (\alpha\alpha')^3$; $^4/_3\ (\gamma t)^4 \equiv (aa')^3\ (\alpha\alpha')\ (\alpha t)^2\ (\alpha' t)^2$; $^4/_3(c\tau)^4 = (\alpha\alpha')^3\ (aa')\ (a\tau)^2\ (a'\tau)^2$. With respect to the original space cubics C_1, C_2 these forms are interpreted as follows. The form $(b\tau)^4\ (\beta t)^4$ vanishes when tangent τ of C_1 meets tangent t of C_2; $\delta = 0$ if the null systems of C_1, C_2 are apolar; $(\gamma t)^4$ or $(c\tau)^4$ vanishes if the tangent t of C_2 or the tangent τ of C_1 is a line of the null system of the other curve.

The reciprocity between the dual forms F, $\overline{F}$ is due algebraically to the fact that their coefficients are, respectively, the one- and three- row minors of A. The four comitants of the second degree above are evidently self-dual in meaning and therefore should involve the two-row minors of A. Indeed we find that the 36 coefficients of the four comitants $(b\tau)^4$ $(\beta t)^4$, $(\gamma t)^4$, $(c\tau)^4$, δ are linearly independent in the 36 two-row minors of A. The only remaining comitants of the second degree, not linear in the minors of A, are the two sextics $S_1(\tau)$, $\bar{S}_2(t)$.

The 36 two-row minors of a four-row determinant A, though linearly independent, must satisfy a system of quadratic relations. It is not hard to see that there are precisely 41 such quadratic relations. If we take the above four comitants and form all of *their* comitants of total second degree in the coefficients of the four we have a set of comitants with 666 coefficients which is the number of quadratic combinations of the 36 minors. Hence the coefficients of this new set of comitants must be connected by a system of 41 linear relations due to the existence of the 41 quadratic relations among the minors. These relations will be expressed by the existence of a system of syzygies of the second degree in the coefficients of the four comitants. We find that this system of syzygies is $(b\tau)^4\ (b'\tau)^4\ (\beta\beta')^4 = 6[(c\tau)^4]^2$; $(bb')^4\ (\beta t)^4\ (\beta' t)^4 = 6[(\gamma t)^4]^2$; $(bb')^2\ (b\tau)^2\ (b'\tau)^2\ (\beta\beta')^4 = 6(cc')^2\ (c\tau)^2\ (c'\tau)^2$; $(bb')^4\ (\beta\beta')^2\ (\beta t)^2\ (\beta' t)^2 = 6(\gamma\gamma')^2\ (\gamma t)^2\ (\gamma' t)^2$; $(b\tau)^4\ (\beta\gamma)^4 = 6\,\delta.(c\tau)^4$; $(bc)^4\ (\beta t)^4 = 6\,\delta.(\gamma t)^4$; $(bb')^4\ (\beta\beta')^4 =$

$$6(\gamma\gamma')^4 = 6(cc')^4 = 36\,\delta^2.$$

The 41 coefficients of these syzygies furnish the quadratic relations. The syzygies themselves determine for given $(b\tau)^4$ $(\beta t)^4$ the three remaining forms $(c\tau)^4$, $(\gamma t)^4$, δ to within a change of sign of any two.

12. *The fundamental combinants of F, $\bar{F}$.* We have had occasion at times to consider such pencils of cubics in t as are determined by the members $(a\tau_1)^3$, $(\alpha t)^3$, $(a\tau_2)^3$ $(\alpha t)^3$ $(\tau_1 \neq \tau_2)$. If t_1, t_2 belong to the same member of this pencil then the fundamental combinant of Gordan for the pencil is

$$\Gamma = \begin{vmatrix} (a\tau_1)^3\ (\alpha t_1)^3 & (a\tau_2)^3\ (\alpha t_1)^3 \\ (a'\tau_1)^3\ (\alpha' t_2)^3 & (a'\tau_2)^3\ (\alpha' t_2)^3 \end{vmatrix}.$$

Evidently it expresses also that τ_1, τ_2 belong to the same member of the pencil determined by the members $(a\tau)^3$ $(\alpha t_1)^3$ $(a\tau)^3$ $(\alpha t_2)^3$ $(t_1 \neq t_2)$. We call Γ the fundamental combinant of F, and $\bar{\Gamma}$ the corresponding fundamental combinant of $\bar{F}$. Each is expressible in terms of the two-row minors of A and therefore in terms of the four comitants of 11. We find that Γ, $\bar{\Gamma} = (\tau_1\tau_2).(t_1t_2) \left\{ (b\tau_1)^2\ (b\tau_2)^2\ (\beta t_1)^2\ (\beta t_2)^2 \pm (t_1t_2)^2.(c\tau_1)^2\ (c\tau_2)^2 \pm (\tau_1\tau_2)^2.\text{-}(\gamma t_1)^2\ (\gamma t_2)^2 + (t_1t_2)^2.(\tau_1\tau_2)^2.\delta \right\}$ where the upper sign holds for Γ and the lower sign (together with the factor $A/9$) for $\bar{\Gamma}$. Let us call the four terms within this brace, taken with plus signs, K, L, M, N, respectively. Then the involution of binary cubics determined in 7 by the form $(\pi x)\ (d\tau)\ (\delta t)^3$ for variable τ has for fundamental combinant (when x is replaced by τ_1, τ_2) $K + L - M - N$

and the corresponding combinant for the counter sextic $S_2(t)$ rather than $\bar{S}_2(t)$ is $K-L+M-N$. These four involutions all arise from any one by changing the signs of any two of the comitants $(\gamma t)^4$, $(c\tau)^4$, δ. In addition to the facts derived in 10 from this set of involutions other results are easily read off from their simple form. For example the conic $\bar{b}$ defined in 7 for $\bar{S}_2(t)$ and the corresponding conic b for $S_2(t)$ are in a pencil with the conic $K(\tau)$ and meet $K(\tau)$ in the four points determined by $(c\tau)^4 = 0$.

LIST OF PUBLICATIONS OF THE NATIONAL ACADEMY OF SCIENCES AND OF THE NATIONAL RESEARCH COUNCIL

PART 1. PUBLICATIONS OF THE NATIONAL ACADEMY OF SCIENCES

From the year 1864 to date the National Academy of Sciences has published at various times Scientific Memoirs, Biographical Memoirs, Annual Reports, Proceedings, and Reports of Committees. All except the Proceedings are distributed free to institutions and students interested in the subjects discussed. The price of subscription to the Proceedings, of which twelve numbers are published each year, is five dollars. The detailed list of titles, with the number of pages and the year of publication, is given below.

MEMOIRS OF THE NATIONAL ACADEMY OF SCIENCES

VOLUME I. 1866

1. Reduction of the observations of fixed stars made by Joseph LePaute D'Agelet, at Paris, in 1783–1785, with a catalogue of the corresponding mean places, referred to the Equinox of 1800.0. BENJAMIN APTHORP GOULD. Read January 8, 1864. Pp. 1–261.
2. The Saturnian system. BENJAMIN PIERCE. Read January 8, 1864. Pp. 263–286.
3. On the distribution of certain diseases in reference to hygienic choice of location for the cure of invalid soldiers. AUGUSTUS A. GOULD. Read August 5, 1864. Pp. 287–290.
4. On shooting stars. H. A. NEWTON. Read August 6, 1864. Pp. 291–312.
5. Rifled guns. W. H. C. BARTLETT. Read August 25, 1865. Pp. 313–343.

VOLUME II. 1884

1. Report of the eclipse expedition to Caroline Island, May, 1883. Pp. 5–146.
2. Experimental determination of wave-lengths in the invisible prismatic spectrum. S. P. LANGLEY. April, 1883. 4 plates. Pp. 147–162.
3. On the subsidence of particles in liquids. WILLIAM H. BREWER. Read November 15, 1883. Pp. 163–175.
4. Upon the formation of a deaf variety of the human race. ALEXANDER GRAHAM BELL. Read November 13, 1883. Pp. 177–262.

VOLUME III. (PART 1.) 1885

1. The sufficiency of terrestrial rotation for the deflection of streams. G. K. GILBERT. Read April 15, 1884. Pp. 5–10.
2. On the temperature of the surface of the moon. From researches made at the Allegheny Observatory by S. P. LANGLEY, assisted by F. W. VERY and J. E. KEELER. Read October 17, 1884. Plates 1–6. Pp. 11–42.
3. On a method of precisely measuring the vibratory periods of tuning-forks, and the determination of the laws of the vibrations of forks; with special reference to these facts and laws to the action of a simple chronoscope. ALFRED M. MAYER. Plates 1–4. Pp. 45–59.

4. The Baumé hydrometers. C. F. Chandler. Read at Philadelphia meeting, 1881. Pp. 61–71.
5. On small differences of sensation. C. S. Peirce and J. Jastrow. Read October 17, 1884. Pp. 73–83.
6. Description of an articulate of doubtful relationship from the Tertiary beds of Florissant, Colorado. Samuel H. Scudder. Read April 20, 1882. Pp. 85–90.
7. The structure of the columella auris in the Pelycosauria. E. D. Cope. Read October 16, 1884. Pp. 93–95.
8. On the structure of the brain of the sessile-eyed Crustacea. I. The Brain of Asellus and the eyeless form Cecidotaea. A. S. Packard. Plates 1–5. Pp. 97–110.

Volume III. (Part 2.) 1886

9. Contributions to meteorology. Elias Loomis. Plates 1–16. Pp. 5–66.
10. On Flamsteed's stars "observed, but not existing." C. H. F. Peters. Read November 11, 1885. Pp. 67–83.
11. Corrigenda in various star catalogues. C. H. F. Peters. Read November 12, 1885. Pp. 85–97.
12. Ratio of meter to yard. C. B. Comstock. Read April 21, 1885. Pp. 99–102.
13. On composite photography as applied to craniology. J. S. Billings. And on measuring the cubic capacity of skulls. Washington Matthews. Read April 22, 1885. 20 Plates. Pp. 103–116.
14. On a new craniphore for use in making composite photographs of skulls. J. S. Billings and Washington Matthews. Read November 12, 1885. 4 plates. Pp. 117–119.
15. I. On the Syncarida, a hitherto undescribed synthetic group of extinct Malacostracous Crustacea. A. S. Packard. Read April 21, 1885. Plates 1–2. Pp. 121–128.
    II. On the Gampsonychidae, an undescribed family of fossil Schizopod Crustacea. A. S. Packard. Read April 21, 1885. Plate 3. Pp. 129–133.
    III. On the Anthracaridae, a family of Carboniferous Macrurous Decapod Crustacea. A. S. Packard. Read April 21, 1885. Plate opposite page 137, and Plate 4. Pp. 135–139.
16. On the Carboniferous Xiphosurous fauna of North America. A. S. Packard. Read November 13, 1885. Plate opposite page 150, and Plates 5–7. Pp. 141–157.
17. On two new forms of Polyodont and Gonorhynchid fishes from the Eocene of the Rocky Mountains. E. D. Cope. Read November 12, 1885. 1 Plate. Pp. 159–165.

Notes on the Third Memoir, Part I, page 45. Alfred M. Mayer. Pp. 167–169.

Volume IV. (Part 1.) 1888

1. The cave fauna of North America, with remarks on the anatomy of the brain and origin of the blind species. A. S. Packard. Read November 9, 1886. Plates 1–27. Pp. 1–156.
2. The solar and lunar spectrum. S. P. Langley. Read November 9, 1886. 6 Plates. Pp. 157–170.
3. On the reduction of photographic observations, with a determination of the position of the Pleiades, from photographs by Mr. Rutherfurd. Benjamin Apthorp Gould. Presented August 11, 1866. Pp. 171–190.

4. Reduction of photographic observations of the Praesepe. BENJAMIN APTHORP GOULD. Presented April 14, 1870. Pp. 191–199.
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VOLUME IV. (PART 2.) 1889

7. Contributions to meteorology. ELIAS LOOMIS. Plates 17–32. Pp. 5–77.
8. On the determination of elliptic orbits from three complete observations. J. WILLARD GIBBS. Pp. 79–104.
9. The temperature of the moon. S. P. LANGLEY. Read November, 1887. Plates 1–26. Pp. 105–212.
10. On the Lucayan Indians. W. K. BROOKS. Read November, 1887. Plates 1–12. Pp. 213–222.

VOLUME V. 1891

1. Energy and vision. S. P. LANGLEY. Presented April 15, 1888. 4 Plates. Pp. 5–18.
2. Contributions to meteorology. ELIAS LOOMIS. Plates 33–51. Pp. 19–109.
3. Report of studies of atmospheric electricity. T. C. MENDENHALL. Read April 18, 1889. Pp. 111–318.
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5. On the application of interference methods to astronomical measurements. A. A. MICHELSON. 7 Plates. Pp. 577–590.

VOLUME VI. 1893

1. On the capture of comets by planets, especially their capture by Jupiter. H. A. NEWTON. Pp. 5–23.
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6. Families and sub-families of fishes. THEODORE GILL. Pp. 125–138.
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8. Further studies on the brain of Limulus Polyphemus, with notes on its embryology. ALPHEUS S. PACKARD. Plates 1–36. Pp. 287–331.

VOLUME VII. 1895

1. Monograph of the Bombycine moths of America, north of Mexico, including their transformations and origin of the larval markings and armature. Part I. Family I. Notodontiadae. ALPHEUS S. PACKARD. Plates 1–49, Maps 1–10. 1895. Pp. 3–390.

2. On reaction-times and the velocity of the nervous impulse. J. McKeen Cattell and Charles S. Dolley. Presented at Albany meeting, 1893. Pp. 391–415.
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Volume VIII. 1902

1. Notes on the bacteriological examination of the soil of Philadelphia. Mazÿck P. Ravenel. 4 Plates. 1896. Pp. 1–41.
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4. Phoronis architecta: Its life history, anatomy, and breeding habits. William Keith Brooks and Reinart Parker Cowels. Plates 1–17. 1905. Pp. 71–148.
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EICHELBERGER, WILLIAM MCKNIGHT RITTER, and G. K. LAWTON. ARMIN O. LEUSCHNER. 1910. Pp. 193–374.

VOLUME XI. 1913

Agave in the West Indies. WILLIAM TRELEASE. Plates A–E, 1–116. 1913. Pp. 3–298.

VOLUME XII. (PART 1.) 1914

1. Monograph of the Bombycine moths of North America, including their transformations and origin of the larval markings and armature. Part III. Families Ceratocampidae (exclusive of Ceratocampinae), Saturniidae, Hemileucidae, and Brahmaeidae. ALPHEUS SPRING PACKARD, edited by THEODORE D. A. COCKERELL, being a continuation of previous investigations published as Volume VII (First Memoir) and Volume IX (Second Memoir) of the Memoirs of the National Academy of Sciences. Plates 1–113. 1914. Pp. 1–516.

VOLUME XII. (PART 2.) 1915

2. The variations and ecological distribution of the snails of the genus Io. CHARLES C. ADAMS. Plates 1–61. 1915. Pp. 1–184.
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1. Catalogue of the meteorites of North America, to January 1, 1909. OLIVER CUMMINGS FARRINGTON. Plates 1–36. 1915. Pp. 1–513.

VOLUME XIV.

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Volume XVII.

1. Biographical memoir of Simon Newcomb. W. W. Campbell. Bibliography of the work of Simon Newcomb. R. C. Archibald. In press.

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Volume I. 1877

Joseph Stillman Hubbard, 1823–1863. B. A. Gould. Read August 5, 1864. Pp. 1–34.

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Benjamin Silliman, Sr., 1779–1864. Alexis Caswell. Read January 25, 1866. Pp. 99–112.

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PROCEEDINGS

Volume I. 1896. 8°. Pp. 1–406. Part 1. Pp. 1–120. Published 1877. Part 2. Pp. 121–240. Published 1886. Part 3. Pp. 241–406. Published 1896.

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Volume 6. 1920. Pp. xiii + 72.

Volume 7. 1921. Pp. vi + 367.

REPORTS OF COMMITTEES

Report on the question of the value of the water-proofing process employed in the manufacture of the fractional currency. In House Misc. Doc. no. 163, part 2, 44th Congress, 1st Session, pp. 22–28. April 3, 1876.

Forty-fifth Congress, 3rd Session, House of Representatives Misc. Doc. no. 5. Surveys of the Territories. Letters from the Acting President of the National Academy of Sciences, transmitting a report on the surveys of the Territories. Ordered printed, December 3, 1878. 8°. Pp. 1–27.

Forty-seventh Congress, 2d Session. Senate Misc. Doc. no. 51. National Academy of Sciences. Investigation of the scientific and economic relations of the sorghum sugar industry, being a report made in response to a request from the Hon. George B. Loring, U. S. Commissioner of Agriculture, by a committee of the National Academy of Sciences. November, 1882. Washington: Government Printing Office. 1883. 8°. Pp. 1–152.

United States Internal Revenue. Report on glucose, prepared by the National Academy of Sciences, in response to a request made by the Commissioner of Internal Revenue. Washington: Government Printing Office. 1884. 8°. Pp. 1–108.

Report of committee of National Academy of Sciences concerning classification of Donskoi wool, January 30, 1886. 1886. Treasury Department Doc. no. 805.

Forty-ninth Congress, 1st Session. Senate, Ex. Doc. no. 67. Letter from the Secretary of the Navy, transmitting, in compliance with the Senate resolution, February 2, 1886, report of the National Academy of Sciences upon the proposed new Naval Observatory. Ordered printed, February 10, 1886.

Report on the organization of the National Surveys and the Signal Service. In Senate Misc. Doc. no. 82, 49th Congress, 1st Session. Pp. 1–37. Ordered printed, March 16, 1886. 1886.

National Academy of Sciences. Standards for electrical measure, February 20,

1895. Printed for the Academy. Washington: Judd & Detweiler, Printers. 1895. 8°. Pp. 1–9.

Fifty-third Congress, 3rd Session. Senate, Misc. Doc. no. 115. Report of the National Academy of Sciences, made in compliance with a requirement of the law (H. R. 6500) entitled "an act to define and establish the units of electrical measure," approved July 12, 1894. Ordered printed, February 19, 1895.

Report of the committee appointed by the National Academy of Sciences upon the inauguration of a forest policy for the forested lands of the United States to the Secretary of the Interior, May 1, 1897. Washington: Government Printing Office. 1897. Pp. 1–47.

Fifty-eighth Congress, 3d Session. Senate Doc. no. 145. Report by committee appointed by Academy to consider desirability of instituting scientific explorations of Philippine Islands. Pp. 1–22. 8°. Ordered printed, February 7, 1905.

Sixtieth Congress, 2d Session. House of Representatives, Doc. no. 1337. Conduct of scientific work under United States Government. Message from the President of the United States, transmitting report of the National Academy of Sciences relating to the conduct of the scientific work under the United States Government. Pp. 1–5. 8°. Ordered printed, January 18, 1909.

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1863	118	1864	1883	145	1884	1902	41	1903
1864	15	1865	1884	69	1885	1903	39	1904
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1866	95	1867	1886	41	1887	1905	41	1906
1867	44	1868	1887	55	1888	1906	43	1907
1868			1888	41	1890	1907	40	1908
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1870			1890	36	1891	1909	43	1910
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1876			1896	39	1897	1915	74	1916
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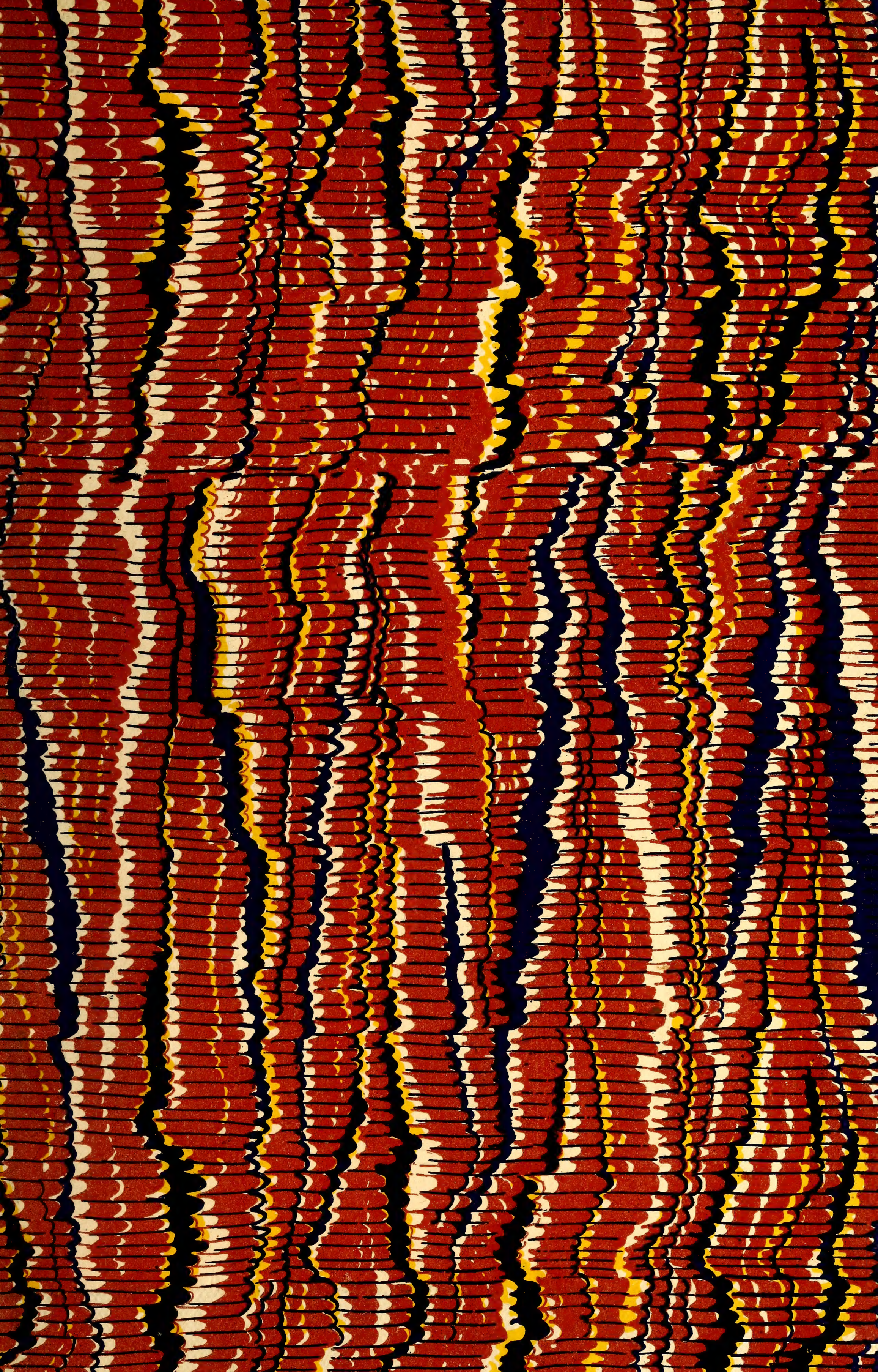

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